# On Toroidal Functions.

XIV. *By* W. M. Hicks, *M.A., Fellow of St. John's College, Cambridge.*
*Communicated by* J. W. L. Glaisher, *M.A., F.R.S.*

Received February 21,—Read March 3, 1881.

The following investigation was originally undertaken as the foundation for certain researches on the theory of vortex rings, with especial reference to a theory of gravitation propounded by the author in the Proceedings of the Cambridge Philosophical Society (vol. iii., p. 276). As the results seemed interesting in themselves, and as they also serve as a basis for other investigations, more particularly in electricity and conduction of heat, I have thought it advisable to publish it as a separate paper, especially as I cannot hope to find leisure for some time to complete my original purpose.

The word "tore" is used as a name for an anchor-ring, here restricted to a circular section, and by "toroidal functions" are understood functions which satisfy Laplace's equation and which are suitable for conditions given over the surfaces of tores.

The *first* section is devoted to the general theory of the employment of two dimensional equipotential lines in certain cases as orthogonal co-ordinates in problems of three dimensions. From this we pass at once to the particular case where the two-dimensional lines are the system of circles through two fixed points and the system of circles orthogonal to them. It is shown that these satisfy the conditions of applicability. By revolution about the line through the two points we have functions suitable for problems connected with two spheres. By revolution about the line bisecting at right angles the distance between the points we have functions associated with anchor-rings or tores. By the first system it is also possible to deduce functions for what may be called a self-intersecting tore, and by the second for two intersecting spheres. A second application is made for the particular case where the opening of a tore vanishes and there is a double cuspidal point at the centre.

The *second* section is devoted to the development of zonal toroidal functions—that is, for conditions symmetrical about the axis* of a tore. It is shown that for space not containing the critical axis these are the same as zonal spherical harmonics of

* Throughout the paper the axis of a tore is taken to be the line perpendicular to its plane through its centre; the circle traced out by the centre of the generating circle of a tore will be called the circular axis, and the circle by the two points above mentioned the critical circle.



The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to
*Philosophical Transactions of the Royal Society of London.* www.jstor.org

imaginary argument and (when the *whole* of space outside a tore is in question) of orders $(2n+1)/2$. For space inside a tore we have a corresponding analogy with zonal harmonics of the second kind. The properties of these functions are found to have analogies with those of the ordinary spherical harmonics, but with essential differences. The space outside a tore is different from that outside a sphere in being cyclic; in general, then, the functions for space outside will not be determinate from the surface conditions alone. The above functions are suitable only when there is no cyclic function: it is shown how to obtain a function which will complete the solution.

The *third* section deals with tesseral toroidal functions, which come into use for the most general case of non-symmetrical conditions. It is shown how the different orders and ranks depend on each other, so that they may be calculated in terms of two. Integral expressions are also obtained, as in the second section, which are needed in finding the coefficients in expansions in series.

The *fourth* section briefly notices the functions suitable for tores without a central opening. These functions bear the same relation to the foregoing functions that cylindric harmonics (BESSEL'S functions) do to spherical harmonics.

In the *fifth* section a few examples are given of the application of the method, such as the potential of a ring, the electric potential of a tore and its capacity, the electric potential of a tore and an electrified circular wire whose axis is the same as that of the tore, the potential under the influence of an electrified point arbitrarily placed, and the velocity potential for a tore moving parallel to its axis, as well as the energy of the motion.

Of previous writings on the subject, or nearly connected therewith, I am only acquainted with two. In RIEMANN'S 'Gesammelte Werke'* (chap. xxiv.) is a short paper of six pages, "Ueber das Potential eines Ringes." He arrives at the same differential equation as (7) in this paper, points out that a solution can be expressed as a hypergeometric series in several ways, and that each function can be expressed in terms of two, which are elliptic integrals of the first and second kinds. The paper is a posthumous one and is not developed. There is a note on the same subject by W. D. NIVEN in the 'Messenger of Mathematics' for December, 1880. Though not bearing on the same subject, a paper may be mentioned by C. NEUMANN, "Allgemeine Lösung des Problemes über den stationären Temperaturzustand eines homogenen Körpers welcher von irgend zwei nichtconcentrischen Kugelflächen begrenzt wird" (H. W. SCHMIDT, Halle). This is a pamphlet of about 150 pages. He uses co-ordinates analogous to those in the present paper, but the method of development is very different. The functions are spherical and cylindric harmonics of real argument, and those of the second kind do not enter. He considers the stationary temperature in a shell bounded by non-concentric spheres; in an infinite medium in which are two spherical cavities; and similar cases when the boundaries touch. Its interest in con-

* The greater portion of the following pages was completed before I became acquainted with this paper of RIEMANN'S or with that of NEUMANN'S mentioned below.

nexion with the following pages consists chiefly in the fact that the potential is expressed in a series of the form

$$f(u.v)\Sigma F_n(u),\ G_n(v)$$

and that the orthogonal co-ordinates employed are closely allied.

## I.

### GENERAL THEORY OF CONJUGATE CURVILINEAR CO-ORDINATES IN THREE DIMENSIONS. SPECIAL CASE.

1. It is well known that if LAPLACE's equation be referred to a system of orthogonal co-ordinates $u$, $v$, $w$ it takes the form

$$\frac{\delta}{\delta u}\left(\frac{U}{VW}\cdot\frac{\delta\phi}{\delta u}\right)+\frac{\delta}{\delta v}\left(\frac{V}{UW}\cdot\frac{\delta\phi}{\delta v}\right)+\frac{\delta}{\delta w}\left(\frac{W}{UV}\frac{\delta\phi}{\delta w}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$U^2=\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2$$

$$V^2=\left(\frac{\delta v}{\delta x}\right)^2+\left(\frac{\delta v}{\delta y}\right)^2+\left(\frac{\delta v}{\delta z}\right)^2$$

$$W^2=\left(\frac{\delta w}{\delta x}\right)^2+\left(\frac{\delta w}{\delta y}\right)^2+\left(\frac{\delta w}{\delta z}\right)^2$$

Let us now take $u$, $v$ to be any conjugate functions of $\rho$, $z$; $\rho$, $z$ being the cylindrical co-ordinates of a point. Also take $w$ to represent a series of planes through the axis of $z$, so that $w=\tan^{-1} y/x$.

Then $u$, $v$, $w$ are orthogonal surfaces and

$$U^2=\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2=\left(\frac{\delta v}{\delta\rho}\right)^2+\left(\frac{\delta v}{\delta z}\right)^2=V^2$$

$$W^2=\frac{1}{\rho^2}$$

So that equation (1) becomes

$$\frac{\delta}{\delta u}\left(\rho\frac{\delta\phi}{\delta u}\right)+\frac{\delta}{\delta v}\left(\rho\frac{\delta\phi}{\delta v}\right)+\frac{1}{\rho\left\{\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2\right\}}\frac{\delta^2\phi}{\delta w^2}=0$$

In this write $\phi=\psi/\sqrt{\rho}$, then

$$\frac{\delta^2\psi}{\delta u^2}+\frac{\delta^2\psi}{\delta v^2}+\frac{\psi}{2\rho^2}\left\{\left|\frac{\delta\rho}{\delta u}\right|^2+\left|\frac{\delta\rho}{\delta v}\right|^2\right\}-\frac{\psi}{2\rho}\left(\frac{\delta^2\rho}{\delta u^2}+\frac{\delta^2\rho}{\delta v^2}\right)+\frac{1}{\rho^2\left\{\left|\frac{\delta u}{\delta\rho}\right|^2+\left|\frac{\delta u}{\delta z}\right|^2\right\}}\frac{\delta^2\psi}{\delta w^2}=0$$

Here since $u$, $v$ are conjugate functions of $\rho$, $z$

$$\frac{\delta^2\rho}{\delta u^2}+\frac{\delta^2\rho}{\delta v^2}=0$$

$$\left(\frac{\delta\rho}{\delta u}\right)^2+\left(\frac{\delta\rho}{\delta v}\right)^2=\frac{1}{\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}=\frac{1}{\xi^2}\text{ (say)}$$

so that

$$\frac{\delta^2\psi}{\delta u^2}+\frac{\delta^2\psi}{\delta v^2}+\frac{1}{\rho^2\xi^2}\left\{\frac{\delta^2\psi}{\delta w^2}+\tfrac{1}{4}\psi\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By putting $\frac{\delta^2\psi}{\delta w^2}=0$ we get the equation for functions satisfying conditions symmetrical about an axis which by an obvious analogy may be called zonal functions. In general, put $\psi=\psi'\cos(mw+\alpha)$, then $\psi'$ must satisfy

$$\frac{\delta^2\psi'}{\delta u^2}+\frac{\delta^2\psi'}{\delta v^2}-\frac{4m^2-1}{4\rho^2\xi^2}\psi'=0$$

When $u$, $v$ are functions such that $1/(\rho\xi)^2$ is of the form $4(f(u)+\mathrm{F}(v))$, it is possible to obtain solutions of the form $\psi=\Sigma\mathrm{X}_{m.n}\mathrm{Y}_{m.n}\cos(mw+\alpha)$ where

$$\frac{d^2\mathrm{X}_{m.n}}{du^2}=(4m^2-1)f(u)\mathrm{X}+n^2\mathrm{X}$$

$$\frac{d^2\mathrm{Y}_{m.n}}{dv^2}=(4m^2-1)\mathrm{F}(v)\mathrm{Y}-n^2\mathrm{Y}$$

which are such that $\mathrm{X}_{m.n}$ are constant when $u$ is constant and $\mathrm{Y}_{m.n}$ constant when $v$ is constant.

As an instance of functions satisfying these conditions we may take the elliptic co-ordinates

$$\rho=a\cosh u\cos v \qquad z=a\sinh u\sin v$$

Here

$$\frac{1}{\rho^2\xi^2}=\frac{1}{\cos^2 v}-\frac{1}{\cosh^2 u}$$

And the equations for the functions X, Y are

$$\frac{d^2\mathrm{X}}{du^2}+\left(\frac{4m^2-1}{4\cosh^2 u}\pm n^2\right)\mathrm{X}=0$$

$$\frac{d^2\mathrm{Y}}{dv^2}-\left(\frac{4m^2-1}{4\cos^2 v}\pm n^2\right)\mathrm{Y}=0$$

The first produces functions analogous to those discussed in this paper—the second spherical harmonics of argument $\frac{\pi}{2}-v$, and order $\frac{m+1}{2}$. The surfaces $u=$ const. give confocal spheroids. Since $\sqrt{\rho}=\sqrt{a\cosh u\cos v}$, it will result that $\phi$ is expressed as the sum of terms of the form $\{AP(u)+BQ(u)\}\{CP'(v)+DQ'(v)\}\cos(mw+\beta)$, where $P'$, $Q'$ are spherical harmonics of argument $\frac{\pi}{2}-v$, and $P$, $Q$ are spherical harmonics of imaginary argument.

In the applications that follow it happens that $u$, $v$ are such that $\rho^2\xi^2$ is a function of $u$ only, say $f(u)$; in this case we obtain solutions by putting $\psi'=\psi\cos(nv+\beta)$, where

$$\frac{d^2\psi}{du^2}-\left(n^2+\frac{4m^2-1}{4f(u)}\right)\psi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The solutions of this equation for $m=0$ may be called zonal functions, for $n=0$ sectorial functions, and for $m.n$ general, tesseral functions. If $U_{m.n}$, $U'_{m.n}$ are two independent solutions of this equation the general value of $\phi$ is given by

$$\sqrt{\rho}\phi=\Sigma\Sigma\{AU_{m.n}\cos(nv+\alpha)\cos(mw+\beta)+A'U'_{m.n}\cos(nv+\alpha')\cos(mw+\beta')\}$$

If $\phi$ be given over any two surfaces $u=$ const., it is clear that the constants can be determined in the above by means of FOURIER'S theorem. This will be more fully discussed in the sequel.

2. Before passing on to particular cases, there is one remarkable result to be noticed. If in the equation transformed as above, we put $\psi=\psi'\cos(\frac{1}{2}w+\gamma)$ then $\psi'$ satisfies the equation

$$\frac{\delta^2\psi'}{\delta u^2}+\frac{\delta^2\psi'}{\delta v^2}=0$$

Hence if $\psi'$ be any two-dimensional potential function, then $\frac{A}{\sqrt{\rho}}\psi'\cos(\frac{1}{2}w+\gamma)$ is a three-dimensional potential function. Since this expression changes sign when $w$ increases by $2\pi$ it is not suitable for all space; but a diaphragm must be supposed to extend from the axis of $z$ to infinity in one direction, and to be impassable. Though the result is interesting it does not seem to carry important consequences, as there is not sufficient generality in the expression. We may choose the form of the surface, and certain other conditions, but all the surface conditions are not arbitrary. Thus let us take an anchor ring divided by a plane through its axis. Let us keep the curved surface and one end at zero temperature, then the distribution of temperature at the other end is determinate though its absolute magnitude is arbitrary. To prove this, we notice that if $(a, b)$ be the radii of the circular axis, and generating circle respectively, and $r$ the distance of any point from the circular axis

$$\psi' = \log\frac{r}{b} = \frac{1}{2}\log\frac{(\rho-a)^2+z^2}{b^2}$$

and

$$t = \frac{A}{2\sqrt{\rho}}\log\frac{(\rho-a)^2+z^2}{b^2}\cos\left(\tfrac{1}{2}w+\gamma\right)$$

This is to be zero when $w=0$ $\therefore$ $\gamma=\frac{\pi}{2}$ and

$$t = \frac{A}{2\sqrt{\rho}}\log\frac{(\rho-a)^2+z^2}{b^2}\sin\frac{w}{2}$$

But now the distribution of temperature at the other end is given by

$$t = \frac{A}{2\sqrt{\rho}}\log\frac{\overline{\rho-a}|^2+z^2}{b^2}$$

leaving only the absolute magnitude A at our disposal. Further, there must be supposed a generation of heat all along the circular axis. This example serves to show the artificiality of solution given by this form.

3. For the case of an anchor ring, or tore, it is at once evident that the proper functions $u$, $v$ to take are the well known ones given by

$$u+vi = \log\frac{\rho+a+zi}{\rho-a+zi}$$

viz.: $v=$const. a series of circles through two points $(\pm a,\ 0)$ and $u=$const. a series of circles orthogonal to them, and each containing one of the fixed points. If these be made to revolve about the line through the fixed points, we get functions proper for two spheres $(u)$; or the surface formed by the revolution of a circle about a line cutting it $(v)$. If they revolve about the axis of $z$, we get functions proper for circular tores $(u)$; or for two intersecting spheres $(v)$. It will be useful to set down here in a compact form, formulæ relating to these functions, which will be required later on. Most of them are easily proved and are set down without proof.

$$\left.\begin{aligned} u &= \tfrac{1}{2}\log\frac{z^2+(\rho+a)^2}{z^2+(\rho-a)^2} \\ v &= -\tan^{-1}\frac{z}{\rho+a}+\tan^{-1}\frac{z}{\rho-a} \\ &= \tan^{-1}\frac{2az}{\rho^2+z^2-a^2} \end{aligned}\right\} \quad \dots\dots\dots \quad (4)$$

$$\left.\begin{aligned}\rho+zi&=a\frac{e^{u+vi}+1}{e^{u+vi}-1}\\ \rho&=a\frac{\sinh u}{\cosh u-\cos v}\\ z&=a\frac{\sin v}{\cosh u-\cos v}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{du}{dn}=\xi=\frac{\cosh u-\cos v}{a}=\frac{\sinh u}{\rho}$$

whence the statement made above that $\rho\xi=f(u)$.

Let R, $r$ be the radii of the circular axis and normal section of a tore $(u)$; $r'$ the radius of a sphere $(v)$; then

$$\left.\begin{aligned}&\mathrm{R}^2-r^2=a^2\\ &\cosh u=\frac{\mathrm{R}}{r}\\ &\sinh u=\frac{\sqrt{\mathrm{R}^2-r^2}}{r}=\frac{a}{r}\\ &\sin v=\frac{a}{r'}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Further, if $r$, the radius of a tore to a point P $(u, v)$ make an angle $\theta$ with the plane of the ring

$$\cos v=-\frac{r-\mathrm{R}\cos\theta}{\mathrm{R}-r\cos\theta}$$

$$\sin v=\frac{\sqrt{\mathrm{R}^2-r^2}\sin\theta}{\mathrm{R}-r\cos\theta}$$

With the above values of $(u, v)$ the general equation for toroidal tesseral functions is

$$\frac{d^2\psi}{du^2}-n^2\psi+\frac{4m^2-1}{4\sinh^2 u}\psi=0 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

There is one case in which the functions used above become nugatory—viz. : when $a$ is zero, or the tores are such that $\mathrm{R}=r$ and they touch themselves at the origin. In this case the proper curves are the two orthogonal families of circles, touching, the one set the axis of $z$, and the other the axis of $\rho$ at the origin—viz. :

$$\left.\begin{aligned} (u+vi)(\rho+zi)&=a \\ u&=\frac{a\rho}{\rho^2+z^2} \\ v&=-\frac{az}{\rho^2+z^2} \\ \rho&=\frac{au}{u^2+v^2} \\ z&=-\frac{av}{u^2+v^2} \\ \frac{du}{dn}=\xi&=\frac{a}{\rho^2+z^2}=\frac{u^2+v^2}{a} \\ &=\frac{u}{\rho} \end{aligned}\right\} \quad . \quad . \quad . \quad (8)$$

and

$$\frac{d^2\psi}{du^2}-n^2\psi-\frac{4m^2-1}{4u^2}\psi=0$$

It will be shown that between the latter surfaces and tores there is a similar relation to that between cylinders and spheres, and between the functions to that between Spherical Harmonics and BESSEL'S functions.

4. The potential due to a ring of radius $b$, centre at $(o.z')$ and plane perpendicular to axis of $z$, is

$$\phi=\int_0^\pi \frac{bd\theta}{\sqrt{(z-z')^2+\rho^2+b^2-2b\rho\cos\theta}}$$

$$=\sqrt{\frac{b}{2\rho}}\int_0^\pi \frac{d\theta}{\sqrt{\alpha-\cos\theta}}$$

In the case where it is the critical circle

$$\alpha=\frac{z^2+\rho^2+a^2}{2a\rho}=\coth u$$

and here

$$\psi=\int_0^\pi \frac{d\theta}{\sqrt{\coth u-\cos\theta}}$$

In general the distance between two points is $(z-z')^2+\rho^2+\rho'^2-2\rho\rho'\cos(w-w')$, which expressed in bipolar co-ordinates becomes

$$\frac{2a^2}{(\cosh u-\cos v)(\cosh u'-\cos v')}\{\cosh u\cosh u'-\cos(v-v')-\sinh u\sinh u'\cos(w-w')\}$$

## II.

## ZONAL TOROIDAL FUNCTIONS.

5. In the case where the conditions are symmetrical about the axis, $\phi$ is independent of $w$ and is of the form

$$\phi=\sqrt{\frac{\cosh u-\cos v}{\sinh u}}\,\Sigma\psi_n\cos(nv+\alpha)$$

where $\psi_n$ is the general integral involving two arbitrary constants of the equation

$$\frac{d^2\psi}{du^2}-n^2\psi+\frac{\psi}{4\sinh^2 u}=0$$

From the potential of a ring, at the end of the last section, it is at once seen that a particular integral for space, not including the critical circle, when $n=0$ is

$$\psi_0=\sqrt{\sinh u}\int_0^\pi\frac{d\theta}{\sqrt{\cosh u-\sinh u\cos\theta}}$$

In the same way, calling the potential of the ring $\phi$, it may be shown by finding $\frac{\delta\phi}{\delta z}$ that

$$\psi_1=\sqrt{\sinh u}\int_0^\pi\frac{d\theta}{(\cosh u-\sinh u\cos\theta)^{\frac{3}{2}}}$$

From analogy with this we might assume

$$\psi_n=\sqrt{\sinh u}\int_0^\pi\frac{d\theta}{(\cosh u-\sinh u\cos\theta)^p}$$

and by substituting we should find it possible by putting $p=\frac{2n+1}{2}$ or $-\frac{2n-1}{2}$ to satisfy the equation. But the following, by making use of theorems already proved for zonal harmonics, seems to be more direct. Putting, in the differential equation,

$$\psi=\sqrt{\sinh u}.\mathrm{P}$$

$$\frac{d^2\mathrm{P}}{du^2}+\coth u\frac{d\mathrm{P}}{du}-(n-\tfrac{1}{2})(n+\tfrac{1}{2})\mathrm{P}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence it is at once evident that $\mathrm{P}_n$ is a zonal spherical harmonic of degree $\frac{2n+1}{2}$, with a pure imaginary for argument. HEINE, in his 'Handbuch der Kugelfunctionen,' has to some extent considered spherical harmonics with imaginary argument, but he has not developed them, at least for fractional indices, in a form suitable for application here. Consequently they will be here considered independently and with especial reference to physical applications. Hereafter, C, S will be used in general to represent cosh $u$, sinh $u$, respectively.

We have then in general

$$\phi=\sqrt{C-\cos v}\,\Sigma(A_nP_n+B_nQ_n)\cos(nv+\alpha)$$

where $P_n$, $Q_n$ are two independent integrals of equation (9). We first discuss the integral already obtained

$$P_n=\int_0^\pi\frac{d\theta}{(C-S\cos\theta)^{\frac{2n+1}{2}}}\quad\ldots\ldots\ldots\ldots\quad(10)$$

It is well known that this integral is the same as

$$\int_0^\pi(C-S\cos\theta)^{\frac{2n-1}{2}}d\theta\quad\ldots\ldots\ldots\ldots\quad(11)$$

the second solution obtained above; this may be easily shown by the transformation $(C-S\cos\theta)(C-S\cos\theta')=1$ or by means of the sequence equation (14) below.

6. *Discussion of* $P_n$.

We have

$$\frac{dP_n}{du}=-\left(\frac{2n+1}{2}\right)\int_0^\pi\frac{S-C\cos\theta}{(C-S\cos\theta)^{\frac{2n+3}{2}}}$$

Whence

$$\frac{2S}{2n+1}\frac{dP_n}{du}=P_{n+1}-CP_n\quad\ldots\ldots\ldots\ldots\quad(12)$$

Similarly from

$$\frac{dP_n}{du}=\frac{2n-1}{2}\int_0^\pi(C-S\cos\theta)^{\frac{2n-3}{2}}(S-C\cos\theta)\delta\theta$$

$$\frac{2S}{2n-1}\frac{dP_n}{du}=CP_n-P_{n-1}\quad\ldots\ldots\ldots\ldots\quad(13)$$

Combining (12) and (13) we have

$$(2n+1)P_{n+1}-4nCP_n+(2n-1)P_{n-1}=0\quad\ldots\ldots\ldots\quad(14)$$

This sequence equation may also be deduced at once from (10) or (11).

In (14) put

$$P_n=\frac{(2n-2)(2n-4)\dots 2}{(2n-1)(2n-3)\dots 3.1}.u_n$$

with

$$P_0=u_0, \quad P_1=u_1$$

then

$$u_{n+2}-2Cu_{n+1}+\frac{(2n+1)^2}{2n(2n+2)}u_n=0$$

or

$$u_{n+2}=2Cu_{n+1}-c_nu_n$$

where

$$c_n=\frac{(2n+1)^2}{2n(2n+2)}=\frac{(2n+1)^2}{(2n+1)^2-1}=1+\tfrac{1}{2}\left(\frac{1}{2n}-\frac{1}{2n+2}\right)$$

and

$$c_0=\tfrac{1}{2}$$

It is clear from this that $u_n$ is of the form $\alpha_nu_1-\beta_nu_0$ where $\alpha_n.\beta_n$ are rational integral algebraical functions of 2C; $\alpha_n$ of degree $n-1$, and $\beta_n$ of degree $n-2$. The first three values are (writing $2C=x$)

$$u_0=u_0$$
$$u_1=u_1$$
$$u_2=xu_1-\tfrac{1}{2}u_0$$
$$u_3=(x^2-c_1)u_1-\tfrac{1}{2}xu_0$$

We can now show that $\alpha_n$, $\beta_n$ are of the form

$$\alpha_n=a_{n.0}x^{n-1}+a_{n.1}x^{n-3}+a_{n.2}x^{n-5}+\dots+a_{n.r}x^{n-2r-1}+\dots$$

For supposing $\alpha_n$ of this form, *i.e.*, wanting every other power of $x$, it follows at once that $\alpha_{n+1}$ is of the same form, and it is seen above that $\alpha_3$ is of this form, whence the statement is generally true, and so also for $\beta_n$.

Now $\alpha_n$ satisfies the equation

$$\alpha_n=x\alpha_{n-1}-c_{n-2}\alpha_{n-2} \quad \dots \dots \dots \dots (15)$$

with

$$\alpha_0=0 \quad \alpha_1=1$$

Hence substituting the above value for $\alpha_n$ we must have

$$a_{n.r}=a_{n-1.r}-c_{n-2}a_{n-2.r-1} \quad \dots \dots \dots \dots (16)$$

also

$$a_{n.0}=a_{n-1.0}=\dots=1$$

Hence

$$a_{n.r} = -(c_{n-2}\, a_{n-2.r-1} + c_{n-3}.\, a_{n-3.r-1} + \dots + c_{2r-1} a_{2r-1.r-1})$$

From this

$$a_{n.1} = -(c_{n-2} + c_{n-3} + \dots + c_1)$$
$$a_{n.2} = \{c_{n-2}(c_{n-4} + \dots + c_1) + c_{n-3}(c_{n-5} + \dots + c_1) + \dots + c_3 c_1$$

= Sum of products two and two together, with the exception of all products where the subscript numbers are successive.

Assume that $(-)^r a_{n.r}$=sum of products of the $c_m$ up to $c_{n-2}$, $r$ together with the exception of any in which successive subscripts occur. Then

$$\begin{aligned} a_{n.r+1} = (-)^{r+1}\{ & c_{n-2}(\text{prod. up to } c_{n-4},\ r \text{ together, \&c.} \dots) \\ & + c_{n-3}(\quad ,, \quad ,, \quad c_{n-5} \dots \quad ) \\ & + \dots \quad \} \\ = (-)^{r+1}\{ & \text{Prod. } (r+1) \text{ together up to } c_{n-2} \text{ without successive subscripts.}\} \end{aligned}$$

Whence by induction the assumption is seen to be universally true. It may be thus stated, $a_{n.r}$ is the sum $r$ together of the terms

$$\frac{3^2}{2.4}, \frac{5^2}{4.6}, \frac{7^2}{6.8}, \dots \frac{(2n-3)^2}{(2n-2)(2n-4)},$$

all products being thrown out in which, regarding the numbers in the denominators as undecomposable, a square occurs in the denominator.

We have

$$a_{n.0} = \quad 1$$
$$a_{n.1} = -\frac{(4n-3)(n-2)}{4(n-1)}$$

This result is of very little use for application. If the co-efficients $(\alpha_n)$ are needed for particular values of $x$ they can be very rapidly calculated by means of equation (15), while if their general values are to be tabulated, equation (16) will serve to calculate them in succession.

Further, $2\beta_n$ is the same kind of function as $\alpha_n$, in every way except that it does not contain $c_1$; in fact $2\beta_n$ is the same function of $c_2, c_3, \dots c_{n-2}$, that $\alpha_{n-1}$ is of $c_1, c_2, \dots c_{n-3}$; calling this $\alpha'_{n-1}$ we can then write

$$u_n=\alpha_n u_1-\tfrac{1}{2}\alpha'_{n-1}u_0$$

$u_1$, $u_0$ may be expressed as elliptic integrals, viz.:

$$\left.\begin{aligned} u_0&=\int_0^\pi\frac{d\theta}{\sqrt{C-S\cos\theta}}=2\sqrt{k'}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=2\sqrt{k'}F \\ u_1&=\int_0^\pi\sqrt{C-S\cos\theta}\,d\theta=\frac{2}{\sqrt{k'}}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\,d\theta=\frac{2}{\sqrt{k'}}E \end{aligned}\right\} \quad . \quad . \quad . \quad (17)$$

where

$$k^2=\frac{2S}{C+S}\qquad k'^2=\frac{1}{(C+S)^2}$$

or

$$k^2=1-e^{-2u}\quad k'^2=e^{-2u}$$

and

$$x=2C=k'+\frac{1}{k'}$$

Hence

$$P_n=2\frac{(2n-2)(2n-4)\ldots 2}{(2n-1)\ldots 3}\left(\frac{\alpha_n}{\sqrt{k'}}E-\tfrac{1}{2}\sqrt{k'}.\alpha'_{n-1}F\right) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where we may suppose the numerical factor dropped if we are dealing with the differential equation, but not if we are dealing with the sequence equation.

The value of $P_n$ when $u=0$ is $\pi$

„ „ $u=\infty$ is $\infty$

These statements are at once seen to be true. Since $u$ becomes infinite along the critical circle it follows that the $P_n$ are not the suitable functions to use by which to express functions which are finite in spaces containing the critical circle, *i.e.*, within any tore. But it is finite and continuous for all space outside any tore.

7. If we put for $P_n$ in equation (9) $P_nQ'_n$ we find in the usual way

$$P_nQ'_n=BP_n+AP_n\int_0^u\frac{du}{S^2P_n^2}$$

Regarded as an analytical solution of the equation this is complete, but in this form it is altogether useless for application. Now HEINE* has shown that the spherical harmonic of the second kind is expressible in the form

$$Q_n(x)=\int_0^\infty\frac{d\theta}{(x-\sqrt{x^2-1}\cosh\theta)^n}$$

* 'Kugelfunctionen,' Kap. iii.

We first show that this with a modification satisfies the equation (9). For putting

$$Q_n=\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{2n+1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{dQ_n}{du}=-\frac{2n+1}{2}\int_0^\infty \frac{S+C\cosh\theta}{(C+S\cosh\theta)^{\frac{2n+3}{2}}}d\theta$$

$$\frac{d^2Q_n}{du^2}=\left(\frac{2n+1}{2}\right)^2 Q_n-\frac{2n+1}{2S}\int_0^\infty \sinh\theta\frac{d}{d\theta}(C+S\cosh\theta)^{-\frac{2n+3}{2}}d\theta$$

$$=\left(\frac{2n+1}{2}\right)^2 Q_n+\frac{2n+1}{2S}\int_0^\infty \frac{\cosh\theta d\theta}{(C+S\cosh\theta)^{\frac{2n+3}{2}}}$$

$$\frac{d^2Q_n}{du^2}+\coth u\frac{dQ}{du}=\left(\frac{2n+1}{2}\right)^2 Q_n-\frac{2n+1}{2S}\int_0^\infty \frac{CS+(C^2-1)\cosh\theta}{(C+S\cosh\theta)^{\frac{2n+3}{2}}}d\theta$$

$$=\frac{4n^2-1}{4}Q_n$$

Here also as in the case of the P functions it can be easily shown that

$$\frac{2S}{2n+1}\frac{dQ_n}{du}=Q_{n+1}-CQ_n$$

$$\frac{2S}{2n-1}\frac{dQ_n}{du}=CQ_n-Q_{n-1}$$

and

$$(2n+1)Q_{n+1}-4nCQ_n+(2n-1)Q_{n-1}=0$$

Hence as before

$$Q_n=\frac{(2n-2)\ldots 2}{(2n-1)\ldots 3}(\alpha_n v_1-\tfrac{1}{2}\alpha'_{n-1}v_0)$$

where

$$v_0=\int_0^\infty \frac{d\theta}{\sqrt{C+S\cosh\theta}}$$

$$v_1=\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{3}{2}}}$$

In these change $\theta$ into $2\theta$, then

$$v_0=2\int_0^\infty \frac{d\theta}{\sqrt{C-S+2S\cosh^2\theta}}$$

$$v_1=2\int_0^\infty \frac{d\theta}{(C-S+2S\cosh^2\theta)^{\frac{3}{2}}}$$

Again, write $\cosh\theta=\sec\phi$, then $\sinh\theta=\tan\phi$, $d\theta=\sec\phi d\phi$, and when $\theta=0$ or $\infty$, $\phi=0$ or $\frac{\pi}{2}$

Hence

$$v_0=2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{C+S-(C-S)\sin^2\phi}}$$

$$=2\sqrt{k'}F' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also

$$v_1=2\int_0^{\frac{\pi}{2}} \frac{\cos^2\phi d\phi}{\{C+S-(C-S)\sin^2\phi\}^{\frac{3}{2}}}$$

$$=\frac{2}{\sqrt{k'}}\int_0^{\frac{\pi}{2}} \frac{k'^2-k'^2\sin^2\phi}{(1-k'^2\sin^2\phi)^{\frac{3}{2}}}d\phi$$

$$=\frac{2F'}{\sqrt{k'}}-\frac{2k^2}{\sqrt{k'}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k'^2\sin^2\phi)^{\frac{3}{2}}}$$

Now

$$k'\frac{dF'}{dk'}=\int_0^{\frac{\pi}{2}} \frac{k'^2\sin^2\phi d\phi}{(1-k'^2\sin^2\phi)^{\frac{3}{2}}}$$

$$=\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k'^2\sin^2\phi)^{\frac{3}{2}}}-F'$$

and

$$k'\frac{dF'}{dk'}=\frac{1}{k^2}E'-F'$$

$$\therefore v_1=\frac{2}{\sqrt{k'}}(F'-E') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and finally

$$Q_n=2\frac{(2n-2)\ldots 2}{(2n-1)\ldots 3}\left\{\frac{\alpha_n}{\sqrt{k'}}(F'-E')-\tfrac{1}{2}\alpha'_{n-1}\sqrt{k'}F'\right\} \quad . \quad . \quad . \quad . \quad (22)$$

The value of $Q_n$ for $u=0$ is $\infty$, and for $u=\infty$ is zero. Hence $Q_n$ is suitable for space within a tore, and not for space including the axis.

8. The foregoing value of $Q_n$ has been obtained from analogy with that for $P_n$; but

in the same way as $P_n$ (for space outside a tore) was obtained from the potential of a ring, so also may $Q_n$ be determined from the potential of a point at the origin, for space not containing it, *i.e.*, for space within a tore. For the inverse distance of a point from the origin is

$$\frac{1}{a}\sqrt{\frac{C-\cos v}{C+\cos v}}$$

Hence

$$\frac{1}{\sqrt{C+\cos v}}=\Sigma(A_nP_n+B_nQ_n)\cos(nv+\alpha)$$

Now, firstly, since this is to be finite throughout all space not including the axis $A_n=0$. Also it is clear since $C>\cos v$ that $1/\sqrt{C+\cos v}$ can be expanded in a series of powers of $\cos v$, and therefore in a series of cosines of multiple angles only. If this be done the coefficient of $\cos nv$ must be $B_nQ_n$.

Hence by FOURIER's theorem

$$\frac{\pi}{2}B_nQ_n=\int_0^\pi\frac{\cos n\theta d\theta}{\sqrt{C+\cos\theta}}$$

If we define $Q_n$ so as to make $\pi B_n/2=(-1)^n\sqrt{2}$, then

$$\sqrt{2}Q_n=(-)^n\int_0^\pi\frac{\cos n\theta d\theta}{\sqrt{C+\cos\theta}}$$

$$=\int_0^\pi\frac{\cos n\theta}{\sqrt{C-\cos\theta}}d\theta. \qquad (23)$$

We will now show that this expression for $Q_n$ agrees with the former one. Integrating by parts and dropping $\sqrt{2}$ as unnecessary in the sequence equation

$$2nQ_n(-)^n=-\int_0^\pi\frac{\sin n\theta\sin\theta}{(C+\cos\theta)^{\frac{3}{2}}}d\theta$$

also

$$(-)^nQ_n=\int_0^\pi\frac{\cos n\theta(C+\cos\theta)}{(C+\cos\theta)^{\frac{3}{2}}}d\theta$$

$$=C\int_0^\pi\frac{\cos n\theta}{(C+\cos\theta)^{\frac{3}{2}}}+\int_0^\pi\frac{\cos n\theta\cos\theta}{(C+\cos\theta)^{\frac{3}{2}}}$$

Hence

$$(2n+1)Q_n(-)^n=\ C\int_0^\pi\frac{\cos n\theta}{(C+\cos\theta)^{\frac{3}{2}}}+\int_0^\pi\frac{\cos(n+1)\theta}{(C+\cos\theta)^{\frac{3}{2}}}$$

$$(2n-1)Q_n(-)^n=-C\int_0^\pi\frac{\cos n\theta}{(C+\cos\theta)^{\frac{3}{2}}}-\int_0^\pi\frac{\cos\overline{n-1}\theta}{(C+\cos\theta)^{\frac{3}{2}}}$$

$$\therefore\ (2n-1)Q_{n-1}+(2n+1)Q_{n+1}=\ (-)^{n+1}C\int_0^\pi \frac{\cos \overline{n-1}\theta-\cos \overline{n+1}\theta}{(C+\cos\theta)^{\frac{3}{2}}}$$

$$=2(-)^{n+1}C\int_0^\pi \frac{\sin n\theta \sin\theta}{(C+\cos\theta)^{\frac{3}{2}}}d\theta$$

$$=4nCQ_n.$$

The same sequence equation as before. Hence it is only necessary further to show that $Q_0$, $Q_1$ are the same in the two cases.

Now

$$Q_0\sqrt{2}=\int_0^\pi \frac{d\theta}{\sqrt{C+\cos\theta}}$$

$$=\int_0^\pi \frac{d\theta}{\sqrt{C+1-2\sin^2\frac{\theta}{2}}}$$

$$=\frac{2}{\sqrt{C+1}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\lambda'^2\sin^2\theta}}$$

where

$$\lambda'^2=\frac{2}{C+1}$$

If C, S be eliminated between $\lambda'$ and $k^2=2S/(C+S)$ there result the equations

$$k=\frac{2\sqrt{\lambda}}{1+\lambda},\quad k'=\frac{1-\lambda}{1+\lambda}$$

Hence by the second quadric transformation

$$F_{\lambda'}=(1+k')F'$$

and since

$$\lambda'=2\sqrt{k'}/(1+k')$$

$$Q_0=2\sqrt{k'}F'.$$

Again

$$-Q_1\sqrt{2}=\lambda'\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta d\theta}{\sqrt{1-\lambda'^2\sin^2\theta}}$$

$$=\frac{\sqrt{2}}{\lambda'}\{2E_{\lambda'}-(1+\lambda^2)F_{\lambda'}\}$$

Now

$$E(\lambda')=\lambda'\lambda^2\frac{dF(\lambda')}{d\lambda'}+\lambda^2F(\lambda')$$

$$=\lambda^2\left\{\lambda'\frac{dk'}{d\lambda'}\frac{d}{dk'}(1+k')F'+(1+k')F'\right\}$$

which on reduction and substitution for $\frac{dF'}{dk'}$ becomes

$$E(\lambda')=\frac{1}{1+k'}(2E'-k^2F')$$

$$\therefore -Q_1\sqrt{2}=\frac{1}{\sqrt{2k'}}\{4E'-2k^2F'-2(1+k'^2)F'\}$$

$$=\frac{2\sqrt{2}}{\sqrt{k'}}(E'-F')$$

and

$$Q_n=2\frac{(2n-2)\ldots 2}{(2n-1)\ldots 3}\left\{\frac{\alpha_n}{\sqrt{k'}}(F'-E')-\tfrac{1}{2}\alpha'_{n-1}\sqrt{k'}F'\right\}$$

We have in fact proved that

$$\int_0^\infty\frac{d\theta}{(\cosh u+\sinh u.\cosh\theta)^{\frac{2n+1}{2}}}=(-1)^n\sqrt{2}\int_0^\pi\frac{\cos n\theta d\theta}{\sqrt{\cosh u+\cos\theta}}$$

By means of the identity

$$EF'+E'F-FF'=\frac{\pi}{2}$$

or

$$E'-F'=\frac{1}{F}\left(\frac{\pi}{2}-EF'\right)$$

$Q_n$ can be expressed in the following manner, viz.:

$$Q_n=2\frac{(2n-2)\ldots 2}{(2n-1)\ldots 3}\left\{\frac{\pi\alpha_n}{2F\sqrt{k'}}-\frac{F'}{F}\left(\frac{\alpha_n E}{\sqrt{k'}}+\tfrac{1}{2}\alpha'_{n-1}F\right)\right\}$$

9. The following relations between P and Q functions will be useful in applications, viz.:

$$\left.\begin{array}{ll}(\alpha) & P_{n+1}Q_n-P_nQ_{n+1}=\dfrac{2\pi}{2n+1}\\[2ex] (\beta) & P'_nQ_n-P_nQ'_n=\dfrac{\pi}{S}\\[2ex] (\gamma) & P'_nQ'_{n+1}-P'_{n+1}Q'_n=(2n+1)\dfrac{\pi}{2}\end{array}\right\}\quad\ldots\ldots\quad(24)$$

They are easily proved, for substituting for $P_{n+1}$, $Q_{n+1}$ from their sequence equations it follows that

$$
\begin{aligned}
(2n+1)(P_{n+1}Q_n-P_nQ_{n+1}) &= (2n-1)(P_nQ_{n-1}-P_{n-1}Q_n) \\
&= P_1Q_0-P_0Q_1 \\
&= 4\{EF'-F(F'-E')\} \\
&= 4(EF'+FE'-FF') \\
&= 2\pi
\end{aligned}
$$

Again

$$
\begin{aligned}
2S(P'_nQ_n-P_nQ'_n) &= (2n+1)\{Q_n(P_{n+1}-CP_n)-P_n(Q_{n+1}-CQ_n)\} \\
&= (2n+1)(P_{n+1}Q_n-P_nQ_{n+1}) \\
&= \pi
\end{aligned}
$$

In a similar way $(\gamma)$ may also be proved.

10. As bearing on the question of the convergency or divergency of series occurring in any investigation it will be important to consider the values of $P_n.Q_n$ when $n$ is infinite. Taking the expression for $P_n$

$$P_n=\int_0^\pi (C-S\cos\theta)^{\frac{2n-1}{2}}d\theta$$

it is clear at once that

$$P_{n+1}<(C+S)P_n>P_n$$

Further since $P_n$ increases with $u$, $\frac{dP}{du}$ is positive, hence $P_{n+1}>CP_n$

Also from

$$Q_n=\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{2n+1}{2}}}$$

$$Q_{n+1}<\frac{1}{C+S}Q_n$$

Also since $Q_n$ decreases with $u$, $\frac{dQ}{du}$ is negative, and therefore $Q_{n-1}>CQ_n$

Hence

$$P_{n+1}Q_{n+1}<P_nQ_n$$

but tends to the limit unity, so that the series

$$\Sigma P_nQ_n \text{ is divergent.}$$

But the series

$$\Sigma P_nQ_n \cos n(v+\alpha) \text{ is convergent,}$$

except when $v+\alpha=0$.

Further if $u' > u$ (P′, Q′, here standing for P($u'$), Q($u'$))

$$P_{n+1}Q'_{n+1} < \frac{C+S}{C'+S'}P_nQ'_n$$

Hence the series

$$\Sigma P_nQ'_n < \Sigma\left(\frac{C+S}{C'+S'}\right)^n$$

and is therefore convergent. Much more then is the series $\Sigma P_nQ'_n \cos u(v+\alpha)$ convergent.

Again if $u' < u$

$$P'_{n+1}Q_{n+1} < \frac{C'+S'}{C+S}P'_nQ_n$$

and as before, the series $\Sigma P'_nQ_n \cos n(v+\alpha)$ is always convergent.

11. Both the functions $P_n.Q_n$, except along the critical circle and axis respectively, make $\phi$ finite, continuous, and single valued when $n$ is integral. The first statement has already been proved, the second follows from the way in which $\frac{dP}{du'}$, $\frac{dQ}{du}$ are expressible in terms of two successive $P_n$ or $Q_n$, and the third is seen to be at once true by integrating $\frac{\delta\phi}{\delta v}$ round a circuit lying on any tore $u=$ constant, when $\int\frac{d\phi}{d\theta}d\theta$ is seen to vanish. Now the space is a cyclic one. Hence the above functions are not suitable for expressing any general conditions in the space without a tore, though they are suitable for any given surface conditions whatever.

Still keeping to physical analogies in order to obtain solutions suitable to this case, we will consider the potential due to a vortex ring or electric current along the critical circle. This would give cyclic functions, but also certain surface conditions. In any particular case then it will be necessary to take account of these surface conditions by means of the $P_n$ or $Q_n$. This potential is measured by the solid angle subtended by the ring.

The (solid angle) $\times\mu$ can be expressed in the form ;—*c.s.* denoting cos $v$, sin $v$,—

$$2\mu\pi - \sqrt{2}\mu \sin v\sqrt{C-c}\int_0^\pi \frac{C+c-S\cos\phi}{s^2+S^2\sin^2\phi}\frac{d\phi}{\sqrt{C-S\cos\phi}}$$

or

$$2\mu\pi - \mu\frac{k\sin v}{\sqrt{S}}\left\{\frac{\sqrt{C+c}+\sqrt{C-c}}{\sqrt{C^2-c^2}-S}\Pi(n_1.k)\right.$$

$$\left.+\frac{\sqrt{C+c}-\sqrt{C-c}}{\sqrt{C^2-c^2}+S}\Pi(n_2.k)\right\}$$

where

$$n_1 = \frac{2S}{\sqrt{C^2-c^2}-S} \qquad n_2 = -\frac{2S}{\sqrt{C^2-c^2}+S}$$

To complete the general expression for $\psi$ we must therefore add a term

$$A \sin v \int_0^\pi \frac{\cosh u + \cos v - \sinh u \cos\phi}{\sin^2 v + \sinh^2 u \, . \, \sin^2 \phi} \frac{d\phi}{\sqrt{\cosh u - \sinh u \cos \phi}} \quad . \quad . \quad . \quad . \quad (25)$$

We shall denote this by the letter $A\Omega$, so that the solid angle varies as $\Omega\sqrt{C-c}$.

## III.

### SECTORIAL AND TESSERAL FUNCTIONS.

12. The differential equation which has to be considered in the general case is

$$\frac{d^2\psi}{du^2} - n^2\psi - \frac{4m^2-1}{4 \sinh {}^2u}\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which in the case of sectorial functions becomes

$$\frac{d^2\psi}{du^2} - \frac{4m^2-1}{4 \sinh {}^2u}\psi = 0$$

In the rest of this paper we shall call $n$ the order of the function and $m$ the rank.

Calling the solution of (7) $\psi_{m.n}$, we proceed to show how $\psi_{m.n}$ can be expressed in terms of $\psi_{m.0}$, $\psi_{m.1}$.

Dropping the $m$ for the time, assume

$$\psi_{n+1} = \psi_n \, . f(u) + \frac{d\psi_n}{du}\phi(u)$$

Then writing $(4m^2-1)/4 = \lambda$, and substituting in the equation which $\psi_{n+1}$ satisfies, making use of the equation for $\psi_n$ to express $\frac{d^2\psi_n}{du^2}$, and $\frac{d^3\psi_n}{du^3}$ in terms of $\psi_n$ and $\frac{d\psi_n}{du}$ we shall get

$$\psi''_{n+1} - (n+1)^2\psi_{n+1} + \frac{\lambda}{S^2}\psi_{n+1}$$
$$= \psi_n\left\{-(2n+1)f + f'' + 2n^2\phi' - \frac{2\lambda}{S}\frac{d}{du}\left(\frac{\phi}{S}\right)\right\} + \psi_n'\{2f' + \phi'' - \overline{2n+1}|\phi\}$$

Now choose $f$, $\phi$, so that

$$\left.\begin{aligned} &f'' - (2n+1)f + 2n^2\phi' - \frac{2\lambda}{S}\frac{d}{du}\left(\frac{\phi}{S}\right) = 0 \\ &\phi'' - (2n+1)\phi + 2f' = 0 \end{aligned}\right\}$$

If we try $\phi=\mathrm{AS}$, we shall find that both the equations

$$\left.\begin{array}{l} f''-(2n+1)f+2n^2\mathrm{AC}=0 \\ 2f'-2n\mathrm{AS}=0 \end{array}\right\}$$

can be satisfied simultaneously if $f=n\mathrm{AC}$.

Hence, whatever $\lambda$ be, the equation

$$\psi_{n+1}=\mathrm{A}\left(n\mathrm{C}\psi_n+\mathrm{S}\frac{d\psi_n}{du}\right)$$

holds.

Again, we may also determine $f$, $\phi$, so that

$$\psi_{n-1}=\psi_n f+\frac{d\psi_n}{du}\phi$$

In this case the equations for $f$ and $\phi$ are

$$\left.\begin{array}{ll} f''+(2n-1)f+2n^2\phi'-\frac{2\lambda}{\mathrm{S}}\frac{d}{du}\left(\frac{\phi}{\mathrm{S}}\right) & =0 \\ \phi''-(2n-1)\phi+2f' & =0 \end{array}\right\}$$

which are satisfied by $\phi=\mathrm{BS}$, $f=-n\mathrm{BC}$.

Hence, $\psi_{n-1}=\mathrm{B}(-n\mathrm{C}\psi_n+\mathrm{S}\psi'_n)$.

The toroidal functions themselves are $\psi/\sqrt{\mathrm{S}}$, and the two particular integrals are represented by $\mathrm{P}_{m.n}$, $\mathrm{Q}_{m.n}$. For these functions the above equations become

$$\mathrm{P}_{m.n+1}=\mathrm{A}_n\{2\mathrm{SP}'_{m.n}+(2n+1)\mathrm{CP}_{m.n}\}$$
$$\mathrm{P}_{m.n-1}=\mathrm{B}_n\{2\mathrm{SP}'_{m.n}-(2n-1)\mathrm{CP}_{m.n}\}$$

and similar equations for the Q.

Since the solutions $\mathrm{P}_{m.n}$ of the differential equation are multiplied by an arbitrary constant, we may, when we confine ourselves to one of the above equations, put A or $\mathrm{B}=1$, and after solving the equation of mixed differences multiply the result by an arbitrary constant. But if we wish to combine both formulæ so as to eliminate the differential co-efficient in them, then the P in both must be the same, and a relation will hold between A and B. This we proceed to find. Dropping the $(m)$ as unnecessary, write

$$\mathrm{P}_n=\mathrm{A}_{n-1}\{2\mathrm{SP}'_{n-1}+(2n-1)\mathrm{CP}_{n-1}\}$$

and substitute therein

$$\mathrm{P}_{n-1}=\mathrm{B}_n\{2\mathrm{SP}'_n-(2n-1)\mathrm{CP}_n\}$$

Whence

$$\text{P}_n = \text{A}_{n-1}\text{B}_n 4\text{S}^2\left(\text{P}''_n + \frac{\text{C}}{\text{S}}\text{P}'_n - \frac{4n^2-1}{4}\text{P}_n - \frac{(2n-1)^2}{4\text{S}^2}\right)$$

which since

$$\text{P}''_n + \frac{\text{C}}{\text{S}}\text{P}'_n - \frac{4n^2-1}{4}\text{P}_n - \frac{m^2}{\text{S}^2}\text{P}_n = 0$$

becomes

$$\text{P}_n = \{4m^2 - (2n-1)^2\}\text{A}_{n-1}\text{B}_n\text{P}_n$$

Hence

$$\text{A}_{n-1}\text{B}_n = \frac{1}{(2m+n-1)(2m-n+1)}$$

If we choose

$$\frac{1}{\text{A}_{n-1}} = 2(m+n) - 1$$

then

$$\frac{1}{\text{B}_n} = 2(m-n) - 1$$

these conditions are satisfied, and the formulæ agree with those found for the zonal function when $m=0$. Hence

$$\left.\begin{aligned} 2\text{SP}'_{m.n} &= (2m+2n+1)\text{P}_{m.n+1} - (2n+1)\text{CP}_{m.n} \\ 2\text{SP}'_{m.n} &= (2m-2n+1)\text{P}_{m.n-1} + (2n-1)\text{CP}_{m.n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From this there follows at once the sequence equation

$$(2m+2n+1)\text{P}_{m.n+1} - 4n\text{CP}_{m.n} + (2n-1-2m)\text{P}_{m.n-1} = 0 \quad . \quad . \quad . \quad (27)$$

In this write

$$\text{P}_{m.n} = \frac{2^{n-1}\underline{|n-1}}{(2m+2n-1)(2m+2n-3)\ldots(2m+1)}u_{m.n}$$

Then

$$u_{m.n+1} - 2\text{C}u_{m.n} + \frac{(2n-1)^2-4m^2}{2n(2n-2)}u_{m.n-1} = 0$$

whence, if

$$c_{m.n} = \frac{(2n-1)^2-4m^2}{(2n-1)^2-1}$$

and $\alpha_{m.n}$, $\alpha'_{m.n-1}$ are the same functions of $c_{m.n}$, &c., as $\alpha_n$, $\alpha'_{n-1}$ are of $c_n$

$$\text{P}_{m.n} = \frac{(2n-2)(2n-4)\ldots 2}{(2m+2n-1)(2m+2n-3)\ldots(2m+1)}\{\alpha_{m.n}\text{P}_{m.1} - \tfrac{1}{2}\alpha'_{m.n-1}\text{P}_{m.0}\} \qquad (28)$$

These formulæ hold for the two particular integrals $\text{P}_{m.n}$ and $\text{Q}_{m.n}$, and they express the tesseral function of any order and rank in terms of sectorial functions and tesseral functions of the first order and same rank. In the same way as was proved in the case of zonal functions, it may be shown that

$$P_{m.n+1}Q_{m.n}-P_{m.n}Q_{m.n+1}$$
$$=\frac{(2n-1-2m)(2n-3-2m)\ldots(1-2m)}{(2n+1+2m)(2n-1+2m)\ldots(3+2m)}(P_{m.1}Q_{m.0}-P_{m.0}Q_{m.1}) \quad . \quad . \quad . \quad (29)$$

also that

$$P'_{m.n}Q_{m.n}-P_{m.n}Q'_{m.n}=\frac{(2n-1-2m)\ldots(1-2m)}{(2n+1+2m)\ldots(3+2m)}\,\frac{P_{m.1}Q_{m.0}-P_{m.0}Q_{m.1}}{2S} \quad . \quad . \quad (30)$$

13. In the same way as relations have been found between successive orders of toroidal functions, relations may be found between successive ranks.

Not putting the order $n$ in evidence, write

$$\psi_{m+1}=f\psi_m+\phi\psi'_m$$

Proceeding as before it will be found that $f$, $\phi$ must satisfy the equations

$$\left.\begin{aligned}&f''-\frac{2m+1}{S^2}f-\frac{4m^2-1}{2}\frac{C}{S^3}\phi+2\left(n^2+\frac{4m^2-1}{4S^3}\right)\phi'=0\\&\phi''-\frac{2m+1}{S^2}\phi+2f'=0\end{aligned}\right\}$$

which are satisfied by

$$\phi=A,\quad f=-\frac{2m+1}{2}A\frac{C}{S}$$

leading to the relations

$$P_{m+1}=A_m\left(P'_m-m\frac{C}{S}P_m\right)$$

In precisely the same manner it may be shown that

$$P_{m-1}=B_m\left(P'_m+m\frac{C}{S}P_m\right)$$

and that

$$A_mB_{m+1}=\frac{4}{4n^2-(2m+1)^2}$$

If we put

$$A_m=\frac{2}{2n+2m+1}$$

then

$$B_m=\frac{2}{2n-2m+1}$$

and when $m=0$ the $P_{0.n}$ have the same values as for the toroidal functions already discussed.

Finally then,

$$\left.\begin{aligned} 2SP'_{m.n}&=\phantom{-}2mCP_{m.n}+(2n+1+2m)SP_{m+1.n} \\ 2SP'_{m.n}&=-2mCP_{m.n}+(2n+1-2m)SP_{m-1.n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

from which the sequence equation follows at once

$$(2m+2n+1)SP_{m+1}+4mCP_m+(2m-2n-1)SP_{m-1}=0\,. \quad . \quad . \quad . \quad (32)$$

If we write in this

$$P_m=\frac{2m(2m-2)\ldots 4.2}{(2m+1+2n)(2m-1+2n)\ldots(1+2n)}\left(\frac{C}{S}\right)^m u_m$$

then

$$u_{m+1}+2u_m+\frac{(2m-1)^2-4n^2}{(2m-1)^2-1}\cdot\left(\frac{S}{C}\right)^2 u_{m-1}=0$$

By combining the formulæ (26) and (31) it is also possible to obtain relations between order and rank together. For instance, from the first of equation (26) and the second of equation (31), we get

$$\begin{aligned}(2m+2n+1)P_{m.n+1}&=(2n+1)CP_{m.n}+2SP'_{m.n} \\ &=(2n+1)CP_{m.n}-2mCP_{m.n}+(2n+1-2m)SP_{m-1.n} \\ &=(2n+1-2m)(CP_{m.n}+SP_{m-1.n})\end{aligned}$$

with three other relations.

The four formulæ are

$$\left.\begin{array}{ll} (26\alpha,\ 31\alpha). & P_{m.n+1}-CP_{m.n}-SP_{m+1.n}=0 \\ (26\alpha.\ 31\beta). & (2n+1+2m)\ P_{m.n+1}-(2n+1-2m)(CP_{m.n}+SP_{m-1.n})=0 \\ (26\beta.\ 31\alpha). & (2m+1-2n)(P_{m.n-1}-CP_{m.n})-(2m+1+2n)SP_{m+1.n}=0 \\ (26\beta.\ 31\beta). & (2m+1-2n)\ P_{m.n-1}+(2m-1+2n)CP_{m.n}+(2m-1-2n)SP_{m-1.n}=0 \end{array}\right\} \ (32a)$$

We are now in a position to reduce still further the relations (29), (30).

For putting $n=0$ in the second of (32a)

$$(2m+1)P_{m.1}=-(2m-1)(CP_{m.0}+SP_{m-1.0})$$

whence

$$(2m+1)\{P_{m.1}Q_{m.0}-P_{m.0}Q_{m.1}\}=(2m-1)S\,(P_{m.0}Q_{m-1.0}-P_{m-1.0}Q_{m.0})$$

But (32)

$$(2m-1)SP_{m.0}=-(2m-3)SP_{m-2.0}-4(m-1)CP_{m-1.0}$$

Therefore the above

$$=S\{P_{1.0}Q_{0.0}-P_{0.0}Q_{1.0}\}$$

But from the first of (32a)

$$S(P_{1.0}Q_{0.0}-P_{0.0}Q_{1.0})=P_{0.1}Q_{0.0}-P_{0.0}Q_{0.1}=2\pi$$

Hence

$$\left.\begin{aligned} P_{m.n+1}Q_{m.n}-P_{m.n}Q_{m.n+1}&=2\frac{(2n-1-2m)(2n-3-2m)\ldots(1-2m)}{(2n+1+2m)(2n-1+2m)\ldots(1+2m)}\cdot\pi \\ \text{and}\qquad P'_{m.n}Q_{m.n}-P_{m.n}Q'_{m.n}&=\frac{(2n-1-2m)\ldots(1-2m)}{(2n+1+2m)\ldots(1+2m)}\cdot\frac{\pi}{S}\end{aligned}\right\}\quad . \quad . \quad (33)$$

In the same way, or by substituting in $P'_{m.n}Q_{m.n}-P_{m.n}Q'_{m.n}$ the first of (26) or the first of (31), there follows

$$P_{m.n+1}Q_{m.n}-P_{m.n}Q_{m.n+1}=S(P_{m+1.n}Q_{m.n}-P_{m.n}Q_{m+1.n})$$

14. From the formulæ now developed it is possible to find the complete integral of the general differential equation. But as in applications the co-efficients are determined in terms of definite integrals it will be well also to consider the solutions from a different point of view. If any potential function be expanded in a series of multiple sines and cosines of $v$, $w$, multiplied by $\sqrt{C-c}$, we know that the co-efficients must be of the form $AP+BQ$. Now such a function is the inverse distance of any point from a fixed point. Let us choose as fixed point, to simplify the expression as much as possible, a point on the axis of $x$ within the critical circle, say $(u'.\pi.0)$. Then the distance of $(u.v.w)$ from this is

$$\frac{a\sqrt{2}}{\sqrt{(C-c)(C'+1)}}\{CC'+c-SS'\cos w\}^{\frac{1}{2}}$$

Hence if $\dfrac{\sqrt{C'+1}}{\sqrt{\{CC'+c-SS'\cos w\}}}$ be expanded in a series, the coefficient of $\cos mw\cos nv$ will be of the form $AP_{m.n}+BQ_{m.n}$. Further, for points within the tore $u'$ (*i.e.*, $u>u'$) $A=0$, whilst for points without, and therefore including the axis $(u<u')$, $B=0$.

Hence

$$\sqrt{C'+1}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos mw\cos nv dw dv}{\sqrt{\{CC'+\cos v-SS'\cos w\}}}=AP_{m.n}\text{ or }BQ_{m.n}$$

according as $u\lessgtr u'$.

Now if the fixed point be on the critical circle B is always equal to zero and $(C'=S'=\infty)$

$$\mathrm{AP}_{m,n}=\int_0^{2\pi}\int_0^{2\pi}\frac{\cos mw \cos nv}{\sqrt{\{\mathrm{C}-\mathrm{S}\cos w\}}}\,dwdv$$

Here A=0 unless $n=0$. Hence

$$\mathrm{AP}_{m,o}=4\pi\int_0^{\pi}\frac{\cos m\theta}{\sqrt{\mathrm{C}-\mathrm{S}\cos\theta}}\,d\theta$$

We have already found that

$$\mathrm{P}_{o,n}\propto\int_0^{\pi}\frac{d\theta}{\{\mathrm{C}-\mathrm{S}\cos\theta\}^{\frac{2n+1}{2}}}$$

we are therefore led to expect that in general

$$\mathrm{P}_{m,n}\propto\int_0^{\pi}\frac{\cos m\theta d\theta}{\{\mathrm{C}-\mathrm{S}\cos\theta\}^{\frac{2n+1}{2}}}$$

which can easily be shown to be the case.

By taking the fixed point at the origin we have

$$\mathrm{BQ}_{m,n}\propto\int_0^{\pi}\int_0^{\pi}\frac{\cos mw \cos nvdwdv}{\sqrt{\mathrm{C}-\cos v}}$$

Here B=0 unless $m=0$, and then

$$\mathrm{Q}_{0.n}\propto\int_0^{\pi}\frac{\cos n\theta}{\sqrt{\mathrm{C}-\cos\theta}}\,d\theta$$

an expression which has been already found.

These expressions as *single* definite integrals are already known to be solutions of the differential equations, and are given by HEINE in his 'Kugelfunctionen.' They may easily be proved directly, and connected with the values found already by the sequence equations, and the values for $\mathrm{P}_{00}$, $\mathrm{P}_{01}$, &c. Thus writing

$$\mathrm{P}_{m.n}=\mathrm{A}\int_0^{\pi}\frac{\cos m\theta d\theta}{(\mathrm{C}-\mathrm{S}\cos\theta)^{\frac{2n+1}{2}}}$$

the integral is easily shown to satisfy equations (32), and the only further condition requisite is that A shall be chosen so as to make it agree with $\mathrm{P}_{o.n}$, $\mathrm{P}_{o.n-1}$.

Now

$$\mathrm{P}_{o.n}=\int_0^{\pi}\frac{d\theta}{(\mathrm{C}-\mathrm{S}\cos\theta)^{\frac{2n+1}{2}}}$$

Hence

$$A=1 \text{ and } P_{m,n}=\int_0^\pi \frac{\cos m\theta d\theta}{(C-S\cos\theta)^{\frac{2n+1}{2}}}$$

Returning to the general integral, since $u$, $u'$ enter symmetrically, and since if $u\gtrless u'$, $u'\lessgtr u$, it follows that

$$\int_0^\pi\int_0^\pi \frac{\cos mw \cos nvdwdv}{\sqrt{CC'-\cos v-SS'\cos w}}=LP_{m,n}.Q'_{m,n} \text{ or } LP'_{m,n}.Q_{m,n}$$

according as $u\lessgtr u'$, where L is independent of $u$ or $u'$. Hence L may be determined by giving particular values to $u$ or $u'$. Suppose $u'$ at the origin then

$$LQ_{m,n}=\lim_{u'=0}\frac{\int_0^\pi\int_0^\pi \frac{\cos mw\cos nvdwdv}{\sqrt{CC'-\cos v-SS'\cos w}}}{\int_0^\pi \frac{\cos m\theta d\theta}{(C'-S'\cos\theta)^{\frac{2n+1}{2}}}}$$

To find the value of this expand the expressions under the integrals in ascending powers of S′, which is ultimately to vanish.

Then if

$$p<m \qquad \int_0^\pi \cos mw \cos^p wdw=0$$

$$p=m \qquad \int_0^\pi \cos mw \cos^m wdw=\frac{\pi}{2^m}$$

$$p>m \qquad \text{the integral is finite} =I \text{ (say)}$$

Hence

$$LQ_{m,n}=\lim\frac{\int_0^\pi \cos nvdv\int_0^\pi \frac{\cos mw}{\sqrt{CC'-\cos v}}\left\{\alpha_m\left(\frac{SS'\cos w}{CC'-\cos v}\right)^m+\ldots\right\}}{\frac{1}{C'^{\frac{2n+1}{2}}}\int_0^\pi \cos m\theta\left(\beta_m\frac{S'^m\cos^m\theta}{C'^m}+\ldots\right)}$$

where

$$\alpha_m= \text{co. of } x^m \text{ in } (1-x)^{-\frac{1}{2}}=\frac{1.3.5\ldots(2m-1)}{2^p\lfloor m}$$

$$\beta_m= \text{,,} \quad \text{,,} \quad (1-x)^{-\frac{2n+1}{2}}=\frac{(2n+1)(2n+3)\ldots(2n+2m-1)}{2^p\lfloor m}$$

Hence

$$LQ_{m,n}=\frac{\alpha_m}{\beta_m}S^m\int_0^\pi \frac{\cos nvdv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

This at once gives us an expression for $Q_{m.n}$, viz.:

$$Q_{m.n}=MS^m\int_0^\pi\frac{\cos nvdv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

where M is some constant depending on $m.n$. This has now to be found. If we write

$$U_{m.n}=S^m\int_0^\pi\frac{\cos nvdv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

it is easily shown that

$$(2n+1-2m)U_{m.n+1}-4nCU_{m.n}+(2n-1+2m)U_{m.n-1}=0$$

This will agree with (27) if

$$NU_{m.n}=\frac{(2m+2n-1)\ldots(2m+1)}{(2n-1-2m)\ldots(1-2m)}Q_{m.n}$$

Hence

$$N=\frac{(2m-1+2n)\ldots(2m+1)}{(2n-1-2m)\ldots(1-2m)}M$$

and

$$Q_{m.o}=NS^m\int\frac{dv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

where N is a function of $m$ only.

Here again this is found to satisfy

$$(2m+1)SQ_{m+1.0}-4mCQ_{m.0}+(2m-1)SQ_{m-1.0}=0$$

which agrees with (32) if

$$N=(-)^mN$$

and

$$Q_{0.0}=N\int_0^\pi\frac{dv}{\sqrt{C-\cos v}}$$

But from the known value of $Q_{0.0}$ we see that $N=1/\sqrt{2}$. Hence

$$Q_{m.n}=\frac{(-)^m}{\sqrt{2}}\frac{(2n-1-2m)\ldots(1-2m)}{(2n-1+2m)\ldots(1+2m)}S^m\int_0^\pi\frac{\cos n\theta d\theta}{(C-\cos\theta)^{\frac{2m+1}{2}}}\quad\ldots\ldots\quad(34)$$

Also

$$L=\frac{\alpha_m}{M\beta_m}$$
$$=(-)^m\sqrt{2}\frac{1.3.5\ldots(2m-1)}{(2n-1+2m)(2n-3+2m)\ldots(2n+1)}\cdot\frac{(2n-1+2m)\ldots(2m+1)}{(2n-1-2m)\ldots(1-2m)}$$
$$=\sqrt{2}L_{m.n}\quad\text{(say)}$$

Since the distance between two points is

$$\frac{a\sqrt{2}}{\sqrt{(C-c)(C'-c')}}\{CC'-\cos(v-v')-SS'\cos(w-w')\}^{\frac{1}{2}}$$

It follows that the potential for a unit point at $(u'.v'.w')$ is

$$\phi=\frac{1}{a}\sqrt{(C-c)(C'-c')}\Sigma\Sigma L_{m.n}P_{m.n}Q'_{m.n}\cos n(v-v')\cos m(w-w')$$

for points outside the tore $u'$; whilst for points inside, it is (35)

$$\phi=\frac{1}{a}\sqrt{(C-c)(C'-c')}\Sigma\Sigma L_{m.n}P'_{m.n}Q_{m.n}\cos n(v-v')\cos m(w-w')$$

where, when $m=0$ or $n=0$, half the above value for $L_{m.n}$ must be taken.

When $u=0$ $P_{m.n}=0$ except when $m=0$ when $P_{0.n}=\pi$, which agrees with the value found in section II.

Also $P_{m.n}$ behaves in a similar way to $P_{0.n}$ for increasing $n$, whilst $P_{m+1.n}>P_{m.n}$ when $m$ is large, as is clear at once from the integral expression for $P_{m.n}$.

Also since

$$Q_{m.n}\propto \dot{S}^m\int_0^\pi\frac{\cos n\theta d\theta}{(C-\cos\theta)^{\frac{2m+1}{2}}}$$

it is clear that when $u=\infty$ $Q_{m.n}=0$ for all values of $m.n$. Also $Q_{m.n}$ behaves as $Q_{0.n}$ for increasing $n$.

## IV.

### TORES WITH NO CENTRAL OPENING.

15. In the case where the hole of a tore vanishes the functions hitherto considered become nugatory. In this case we must have recourse to the co-ordinates already referred to in (8). It is not here intended to develop the theory with the fulness of the general case. The functional differential equation has been shown to be

$$\frac{d^2\psi}{du^2}-n^2\psi-\frac{4m^2-1}{4u^2}\psi=0$$

In this write $\psi=\sqrt{u}G$ when

$$\frac{d^2G}{du^2}+\frac{1}{u}\frac{dG}{du}-n^2G-\frac{m^2}{u^2}G=0$$

the equation of cylindric harmonics (BESSEL'S functions) with imaginary argument. Let G.H be the two particular integrals corresponding to the cylindric function J.Y of the first and second kind. Then

$$G_m(nu)=J_m(nui)$$
$$H_m(nu)=Y_m(nui)$$

And the potential function can be expressed in the form

$$\phi=\sqrt{u^2+v^2}\Sigma\Sigma(A_mG_m(nu)+B_mH_m(nu))\cos(nv+\alpha)\cos(mw+\beta)$$

Many of the properties of these functions can be at once written down from the analogous properties of J.Y. Thus

$$G_m(nu)=\frac{(nu)^m}{2.4\ldots 2m}\left\{1+\frac{n^2u^2}{2.2m+2}+\frac{n^4u^4}{2.4.(2m+2)(2m+4)}+\ldots\right\}$$
$$=\frac{1}{\pi}\int_0^\pi\cos(nu\sin\theta-m\theta)d\theta$$
$$=\frac{(nu)^m}{1.3.5\ldots 2m-1}\cdot\frac{1}{\pi}\int_0^\pi\cos(nz\cos\theta)\sin^{2m}\theta d\theta$$

&c.

So also

$$\left.\begin{aligned}u\frac{dG_m}{du}&=mG_m-uG_{m+1}\\&=uG_{m-1}-mG_m\\G_{m+1}-\frac{2m}{u}G_m+G_{m-1}&=0\end{aligned}\right\}$$

and

which equations the H also satisfy.

The sequence equation has been solved by LOMMELL,* so as to fully express $G_m$ and $H_m$ in terms of $G_0$, $G_1$, $H_0$, $H_1$. But in any particular case where the values are required it is best to calculate successively by means of the sequence equation direct.

In the space within a tore $u$ can become infinite, viz.: at the origin, and is never zero; this is evident from the equation

$$u=\frac{a\rho}{\rho^2+z^2}$$

* 'Studien über die Bessel'schen Functionen,' p. 4.

Without, it may become zero along the axis but infinite nowhere, for as it approaches the origin $u$ must approach a finite limit which depends on the circle along which it moves. Now when $u$ is infinite, G is infinite. Hence the G functions belong to space outside a tore. We are led to conclude that the H functions belong to space within. This may be proved as follows: Amongst many integral expressions known for $Y_0$ one is given by HEINE,* viz.:

$$\int_0^\infty e^{ix\cosh\theta}d\theta$$

This suggests

$$H_0(u)=A\int_0^\infty e^{-u\cosh\theta}d\theta$$

This is easily verified, for substituting in the differential equation it has to be shown that

$$\int_0^\infty (\sinh^2\theta-\frac{1}{u}\cosh\theta)e^{-u\cosh\theta}d\theta=0$$

which, on integrating the first term by parts follows at once. From this form we gather that

$$\text{when } u=0 \qquad H_0=\int_0^\infty d\theta=\infty$$

$$\text{,,} \quad u=\infty \qquad H_0=0$$

whence $H_0$ is the proper function for space within a tore. From the sequence equation this is seen to apply also to the $H_m$ in general.

## V.

### EXAMPLES AND APPLICATIONS.

In this section I propose to give a few examples of the application of the foregoing theory, to the solution of physical problems.

16. *Potential of a ring whose axis is the same as the critical circle.*

Let $z'$ be its distance from the plane of the critical circle, $b$ its radius, $u'$, $v'$ its dipolar co-ordinates.

Then the potential is

$$2\mu b\int_0^\pi \frac{d\theta}{\sqrt{(z-z')^2+\rho^2+b^2-2b\rho\cos\theta}}$$

This expanded takes the form

$$\sqrt{C-c}\Sigma A_n P_n \cos(nv+\alpha_n)$$

for points outside the tore $u'$.

* 'Kugelfunctionen,' p. 191.

For a point on the axis, $u=0$ $P_n=\pi$ and the above become respectively

$$\frac{2\mu\pi\sqrt{1-c}.S'}{\sqrt{2(C'-\cos\overline{v-v'})}} \text{ and } \pi\sqrt{1-c}\Sigma A_n\cos(nv+\alpha_n).$$

It will therefore be more convenient to determine the $A_n$, $\alpha_n$ from this simplified case.

It is clear that $1/\{C'-\cos\overline{v-v'}\}^{\frac{1}{2}}$ can be expanded in a series of powers of cosines of $(v-v')$ and therefore of multiples of the same.

Hence

$$\alpha_n=-nv'$$

and

$$\frac{2\mu S'}{\sqrt{2(C'-\cos\theta)}}=\Sigma A_n\cos n\theta$$

Therefore

$$\pi A_n=\frac{2\mu S'}{\sqrt{2}}\int_0^{2\pi}\frac{\cos n\theta d\theta}{\sqrt{C'-\cos\theta}}$$
$$=4\mu S'Q'_n$$

But

$$\pi A_0=2\mu S'Q'_0$$

Hence in general the potential for points outside the tore $u'$ is

$$\phi=\frac{4\mu S'}{\pi}\left(\frac{C-c}{C'-c'}\right)^{\frac{1}{2}}\{\Sigma P_nQ'_n\cos n(v-v')-\tfrac{1}{2}P_0Q'_0\} \quad . \quad . \quad . \quad . \quad . \quad (36a)$$

Consequently the potential for points within the tore $u'$ is

$$\phi=\frac{4\mu S'}{\pi}\left(\frac{C-c}{C'-c'}\right)^{\frac{1}{2}}\{\Sigma P'_nQ_n\cos n(v-v')-\tfrac{1}{2}P'_0Q_0\} \quad . \quad . \quad . \quad . \quad . \quad (36b)$$

Both these series have been shown to be convergent.

If M be the whole mass of the ring

$$M=2\pi b\mu=2\pi\mu\frac{aS'}{C'-c'}$$

It follows as a corollary that the potential for a mass M on the axis is, for all points not on the axis,

$$\frac{2M\sqrt{2}\sin\frac{v'}{2}}{a\pi}\sqrt{C-c}\{\Sigma_0^\infty Q_n\cos n(v-v')-\tfrac{1}{2}Q_0\} \quad . \quad . \quad . \quad . \quad (37)$$

Also, putting M at $0.v'$ and $-M$ at $0.-v'$, and making $v'$ zero and M infinite, the potential for a uniform field of force parallel to axis is

$$\mu\sqrt{C-c}\Sigma_0^\infty nQ_n\sin nv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

17. *Electric potential of a tore and its capacity.*

Let V be the constant potential of the tore ($u'$). Then ($A_n$, $\alpha_n$) must be determined, so that

$$\phi=\sqrt{C-c}\Sigma_0^\infty A_n P_n \cos n(v+\alpha_n)$$

may$=$V for all values of $v$ when $u=u'$.

Hence $\alpha_n=0$ and

$$\pi A_n P'_n=2V\int_0^\pi \frac{\cos n\theta d\theta}{\sqrt{C'-\cos\theta}}$$
$$=2\sqrt{2}VQ'_n$$

and

$$\pi A_0 P'_0=\sqrt{2}VQ'_0$$

$$\therefore \phi=\frac{\sqrt{2}V}{\pi}\sqrt{C-c}\left\{2\Sigma_1\frac{Q'_n}{P'_n}P_n\cos nv+\frac{Q'_0}{P'_0}P_0\right\} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

This series is easily seen to be convergent, since (§ 10) it is less than $\Sigma\frac{(C+S)^n}{(C'+S')^{2n}}$, where $C+S<C'+S'$.

To find the capacity of the ring we must take the surface integral of $\frac{1}{4\pi V}\frac{\delta\phi}{\delta n}$ over it. So, $q$ denoting the capacity,

$$q=\frac{1}{4\pi V}\int_0^{2\pi}2\pi\rho\frac{dn'}{dv}dv\frac{\delta\phi}{\delta u}\cdot\frac{du}{dn}$$
$$=\frac{\sqrt{2}aS}{\pi}\Sigma\int_0^{2\pi}\frac{1}{C-c}\left\{\frac{S}{2\sqrt{C-c}}P_n+\sqrt{C-c}\frac{dP_n}{du}\right\}\frac{Q'_n}{P'_n}\cos nvdv$$

or dropping the dashes, and writing

$$\frac{dP_n}{du}=\frac{2n+1}{2S}(P_{n+1}-CP_n)$$

$$q=\frac{2\sqrt{2}aSQ_n}{\pi}\int_0^\pi\left\{-\frac{d}{du}+\frac{2n+1}{2S}\left(\frac{P_{n+1}}{P_n}-C\right)\right\}\frac{\cos nv}{\sqrt{C-c}}dv$$

Now

$$\int_0^\pi\frac{\cos nvdv}{\sqrt{C-c}}=\sqrt{2}Q_n$$

and

$$\frac{dQ_n}{du}=\frac{2n+1}{2S}(Q_{n+1}-CQ_n)$$

whence

$$q=\frac{2a}{\pi}\Sigma(2n+1)(P_{n+1}Q_n-P_nQ_{n+1})\frac{Q_n}{P_n}$$
$$=4a\Sigma_1\frac{Q_n}{P_n}+2a\frac{Q_0}{P_0}\text{ by (24)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This expression for the capacity in an infinite series is more convergent than $\Sigma e^{-2nu}$.

When the section of the ring is not very large compared with the radius of its circular axis,

$$q=2a\left(\frac{Q_0}{P_0}+2\frac{Q_1}{P_1}\right) \text{ very nearly*}$$

or

$$\left.\begin{aligned} &=2a\left(\frac{F'}{F}+2\frac{F'-E'}{E}\right) \\ &=2\sqrt{R^2-r^2}\left\{3\frac{F'}{F}-\frac{\pi}{EF}\right\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40a)$$

where

$$k^2=2\frac{\sqrt{R^2-r^2}}{R+\sqrt{R^2-r^2}}$$

Measured in terms of the capacity of a sphere whose radius is equal to a tangent from the centre to the tore, the capacity is

$$2\frac{3EF'-\pi}{EF}$$

When $R=3r$ the omission of the term depending on $\frac{Q_2}{P_2}$ introduces an error of about ·27 per cent.

$k^2$ may be expressed in terms of the angle subtended at the centre by the tore, viz.: if this angle be $2\alpha$,

$$k^2=\frac{2\cos\alpha}{1+\cos\alpha}=\cos\alpha\sec^2\frac{\alpha}{2}$$

$$k'=\tan\frac{\alpha}{2}$$

When

$k'=\sin 3^0$ (about $r=\frac{1}{10}R$) $\qquad q=·733\times$ capacity of above sphere

$k'=\sin 6^0$ (about $r=\frac{1}{5}R$) $\qquad q=·898\times$ . . .

18. We may find the potential also for the electricity induced on a tore, put to earth, by a charged circular wire with the same axis as the tore. For the potential of the wire $(u', v')$ for points within $(u')$ is (36b)

$$\phi_1=\frac{4\mu S'}{\pi}\sqrt{C-c}\{\Sigma P'_nQ_n\cos n(v-v')-\tfrac{1}{2}P'_0Q_0\}$$

whilst that for points outside the tore $(u_0)$ due to the charge induced on it is

$$\phi_2=\sqrt{C-c}\Sigma A_nP_n'\cos n(v-\alpha_n)$$

* The expression given in the 'Proceedings' is incorrect.

and the condition is that when $u=u_0$ $\phi_1+\phi_2=0$

$$\therefore\ \alpha_n=v',\ \mathrm{A}_n\mathrm{P}^0_n=-\frac{4\mu\mathrm{S}'}{\pi}\mathrm{P}'_n\mathrm{Q}^0_n,\ \text{and}\ \mathrm{A}_0\mathrm{P}^0_0=-\frac{2\mu\mathrm{S}'}{\pi}\mathrm{P}'_0\mathrm{Q}^0_0$$

Whence

$$\phi=\frac{2\mu\mathrm{S}'}{\pi}\sqrt{\mathrm{C}-c}\,\Sigma\left\{2\frac{\mathrm{P}'_n}{\mathrm{P}^0_n}(\mathrm{P}^0_n\mathrm{Q}_n-\mathrm{Q}^0_n\mathrm{P}_n)\cos n(v-v')+\frac{\mathrm{P}'_0}{\mathrm{P}^0_0}(\mathrm{P}^0_0\mathrm{Q}_0-\mathrm{Q}^0_0\mathrm{P}_0)\right\}\quad . \quad (41)$$

and the general solution when the tore is insulated and has a charge of its own is found by adding the potential found in the last article.

Also if the section of the wire be very small we can find the capacity of the system approximately, by supposing the wire to coincide with one of the equipotential surfaces near it.

19. As an example of the use of tesseral functions with constant surface conditions, we will take the problem of the electrical induction on a tore under the influence of a point arbitrarily placed. We lose no generality by supposing it in the plane of $(xz)$; let then its co-ordinates be $(u'.v'.0)$. The potential due to this for points within $u'$ has been found at the end of Section III., viz.,

$$\phi=\frac{\mu}{a}\sqrt{(\mathrm{C}-c)(\mathrm{C}'-c')}\ \Sigma\mathrm{L}_{m.n}\mathrm{P}'_{m.n}\mathrm{Q}_{m.n}\cos mw\cos n(v-v')$$

As before, the potential of the induced charge will be of the form

$$\phi=\sqrt{\mathrm{C}-c}\ \Sigma\mathrm{A}_{m.n}\mathrm{P}_{m.n}\cos mw\cos n(v-v')$$

and (the tore being $u_0$)

$$\mathrm{A}_{m.n}\mathrm{P}^0_{m.n}=-\mu\frac{\sqrt{\mathrm{C}'-c'}}{a}\mathrm{L}_{m.n}\mathrm{P}'_{m.n}\mathrm{Q}^0_{m.n}$$

$$\phi=\mu\frac{\sqrt{(\mathrm{C}-c)(\mathrm{C}'-c')}}{a}\Sigma\mathrm{L}_{m.n}\frac{\mathrm{P}'_{m.n}}{\mathrm{P}^0_{m.n}}(\mathrm{P}^0_{m.n}\mathrm{Q}_{m.n}-\mathrm{Q}^0_{m.n}\mathrm{P}_{m.n})\cos mw\cos n(v-v')\ . \quad (42)$$

When the point is on the axis, all these terms vanish (§ 14) except for $m=0$.

If necessary, also, the capacity of a tore and a very small sphere can be found approximately from these formulæ.

20. One more example illustrating the application to cases of differential surface conditions may be given. Take the case of a tore moving parallel to its axis through an infinite fluid with velocity V. Here the conditions are that if $\phi$ be the velocity potential for fluid moving past it,

$$\phi=-\mathrm{V}z+\phi_1$$

and

$$\frac{\delta\phi}{\delta u}=0\ \text{when}\ u=u_0.$$

The expansion for $z$ has already been given, viz.: (for points not on the axis)

$$\mu\sqrt{\mathrm{C}-c}\,\Sigma_0^\infty n\mathrm{Q}_n \sin nv$$

To determine $\mu$ we notice that at the critical circle (as everywhere on the plane of $xy$)

$$\frac{d\phi}{dv}\frac{dv}{dn}=-\mathrm{V}$$

Taking a point outside the critical circle

$$-\mathrm{V}=\mu\frac{(\mathrm{C}-1)^{\frac{3}{2}}}{a}\Sigma n^2\mathrm{Q}_n$$

The easiest way to calculate this is to make the point approach the critical circle, *i.e.*, $u=\infty$, when

$$-\mathrm{V}=\frac{\mu}{a}\lim\,(\mathrm{C}-1)^{\frac{3}{2}}\mathrm{Q}_1$$

$$=\frac{\mu}{a}\lim\,(\mathrm{C}-1)^{\frac{3}{2}}\int_0^\infty\frac{d\theta}{(\mathrm{C}+\mathrm{S}\cosh\theta)^{\frac{3}{2}}}$$

$$=\frac{\mu}{a}\int_0^\infty\frac{d\theta}{2^{\frac{3}{2}}\cosh^3\frac{\theta}{2}}=\frac{\mu}{a}\cdot\frac{\pi}{4\sqrt{2}}$$

which gives the theorem

$$\Sigma n^2\mathrm{Q}_n=\frac{\pi}{4\sqrt{2}}(\mathrm{C}-1)^{-\frac{3}{2}}=\frac{\pi}{16}\,\mathrm{cosech}^3\,\frac{u}{2}$$

Hence

$$\phi=\frac{4a\sqrt{2}}{\pi}\mathrm{V}\sqrt{\mathrm{C}-c}\,\Sigma(\mathrm{A}_n\mathrm{P}_n\sin n(v-\alpha_n)-n\mathrm{Q}_n\sin nv)$$

where $\frac{\delta\phi}{\delta u}=0$ when $u=u_0$ for all values of $v$. The terms in cos $nv$ would merely increase $\phi$ by the series for a constant, we may therefore without loss of generality put $\alpha_n=0$, and then, using dashed letters to denote differential coefficients,

$$\frac{1}{4\sqrt{2}}\,\frac{\pi}{a\mathrm{V}}\,\frac{\delta\phi}{\delta u}=\Sigma\left\{\frac{\mathrm{S}}{2\sqrt{\mathrm{C}-c}}(\mathrm{A}_n\mathrm{P}_n-n\mathrm{Q}_n)+\sqrt{\mathrm{C}-c}(\mathrm{A}_n\mathrm{P}'_n-n\mathrm{Q}'_n)\right\}\sin nv$$

$$=0 \text{ when } u=u_0$$

$$\therefore\ \Sigma\{\mathrm{S}(\mathrm{A}_n\mathrm{P}_n-n\mathrm{Q}_n)+2(\mathrm{C}-c)(\mathrm{A}_n\mathrm{P}'_n-n\mathrm{Q}'_n)\}\sin nv=0$$

$$\mathrm{A}_{n+1}\mathrm{P}'_{n+1}+\mathrm{A}'_{n-1}\mathrm{P}'_{n-1}-\mathrm{A}_n(\mathrm{SP}_n+2\mathrm{CP}'_n)$$

$$=(n+1)\mathrm{Q}'_{n+1}+(n-1)\mathrm{Q}'_{n-1}-n(\mathrm{SQ}_n+2\mathrm{CQ}'_n$$

Now it is easily shown that

$$P'_{n+1}+P'_{n-1}-(SP_n+2CP'_n)=0$$

with a similar formula for $Q_n$, we may hence write the above equation

$$(A_{n+1}-A_n)P'_{n+1}-(A_n-A_{n-1})P'_{n-1}=Q'_{n+1}-Q'_{n-1}$$

with initial equation

$$(A_2-A_1)P'_2-A_1P'_0=Q'_2-Q'_0$$

To get a first integral of this write the successive equations in order, multiply those containing $P'_{r+1}$, $P'_{r-1}$ by $P'_r$ and add, we get

$$(A_{n+1}-A_n)P'_{n+1}P'_n-A_1P'_1P'_0=P'_nQ'_{n+1}-P'_1Q'_0+\Sigma_1^{n-1}(P'_rQ'_{r+1}-P'_{r+1}Q'_r)$$

$$=P'_nQ'_{n+1}-P'_0Q'_1+\frac{\pi}{2}\Sigma_0^{n-1}(2r+1)$$

$$\therefore\ A_{n+1}-A_n=\frac{Q'_{n+1}}{P'_{n+1}}+\frac{(A_1P'_1-Q'_1)P'_0}{P'_nP'_{n+1}}+\frac{\pi}{2}\cdot\frac{n^2}{P'_nP'_{n+1}}$$

Put

$$(A_1P'_1-Q'_1)P'_0=\frac{\pi}{2}\alpha.,\quad \frac{Q'_n}{P'_n}=x_n,$$

then since

$$\frac{\pi}{2}=\frac{1}{2n+1}(P'_nQ'_{n+1}-P'_{n+1}Q'_n)$$

$$A_{n+1}-A_n=x_{n+1}+\frac{n^2+\alpha}{2n+1}(x_{n+1}-x_n)$$

$$=\frac{1}{2n+1}\{(\overline{n+1}|^2+\alpha)x_{n+1}-(n^2+\alpha)x_n\}$$

Hence

$$A_{n+1}-A_1=\frac{(n+1)^2+\alpha}{2n+1}x_{n+1}+2\Sigma_2^n\frac{r^2+\alpha}{4r^2-1}x_r-\frac{1+\alpha}{3}x_1$$

and

$$A_{n+1}=\frac{(n+1)^2+\alpha}{2n+1}x_{n+1}+2\Sigma_1^n\frac{r^2+\alpha}{4r^2-1}x_r-\alpha x_0$$

$A_n$ is undetermined to the extent of $\alpha$; but since the velocity potential must be finite everywhere, $\alpha$ must be chosen so that the series $\Sigma A_nP_n(u)$ shall be convergent. It will first be necessary to prove that $A_n$ is finite when $n$ is large; $\alpha$ must then be chosen so that $A_n$ vanishes for $n$ infinite, and lastly, it will remain to show that with this value of $\alpha$ the series $\Sigma A_n P_n(u)$ is convergent, from which the convergency of $\phi$ will flow at once. Now

$$-x_n=-\frac{Q'_n}{P'_n}=\frac{Q_{n-1}-CQ_n}{CP_n-P_{n-1}}$$

$$<\frac{Q_{n-1}-CQ_n}{S^2P_{n-1}}$$

$$\therefore -\Sigma_1^\infty x_n<\frac{1}{S^2}\Sigma_1^\infty\frac{Q_{n-1}}{P_{n-1}}-\frac{C}{S^2}\Sigma_1^\infty\frac{Q_n}{P_{n-1}}$$

Both the series on the right are finite, hence so also are $-\Sigma_1^\infty x_r$ and $\Sigma_1^\infty\frac{r^2+\alpha}{4r^2-1}x_r$, and $A_n$ tends to a finite limit with increasing $n$. It is therefore possible to give $\alpha$ a value which shall make this limit zero. It is given by

$$2\Sigma_1^\infty\frac{r^2+\alpha}{4r^2-1}x_r-\alpha x_0=0$$

whence

$$A_n=\frac{n^2+\alpha}{2n-1}x_n-2\Sigma_n^\infty\frac{r^2+\alpha}{4r^2-1}x_r \quad \dots\dots\dots \quad (43)$$

Lastly it remains to consider the convergency of the series $\Sigma A_n P_n(u)$. When $n$ is very large $-A_{n+1}$ tends to the limit $-\frac{(n+1)^2+\alpha}{2n+1}x_{n+1}$ which is

$$<\frac{(n+1)^2+\alpha}{(2n+1)S^2}\left(\frac{Q_{n-1}}{P_{n-1}}-C\frac{Q_n}{P_{n-1}}\right)$$

Also since $u<u_0$ $P_n(u)<P_n$. Hence the series under consideration is

$$<\frac{1}{S^2}\Sigma\frac{n^2+\alpha}{(2n-1)}\left\{\frac{Q_{n-2}P_n}{P_{n-2}}-C\frac{Q_{n-1}P_n}{P_{n-2}}\right\}$$

The sum of the first set of terms is $<\frac{(C+S)^2}{S^2}\Sigma\frac{n^2+\alpha}{2n-1}Q_{n-2}$, and of the second set is $<\frac{C(C+S)^2}{S^2}\Sigma\frac{n^2+\alpha}{2n-1}Q_{n-1}$ ; both of these are finite. Hence the sum $\Sigma A_n P_n$ is finite and *à fortiori* the sum $\Sigma A_n P_n(u)\sin nv$.

Finally then the velocity potential for fluid motion due to a tore moving parallel to itself through a fluid at rest at infinity is

$$\phi=\frac{4a\sqrt{2}}{\pi}V\sqrt{C-c}\,\Sigma_1^\infty A_n P_n \sin nv$$

where $A_n$ is given by (43) and

$$\alpha=\frac{-2\Sigma_1^\infty\frac{r^2}{4r^2-1}\frac{Q'_r}{P'_r}}{-x_0+2\Sigma_1^\infty\frac{1}{4r^2-1}\cdot\frac{Q'_r}{P'_r}}$$

and where $P'_r \, . \, Q'_r$ stand for $\frac{dP_r(u_0)}{du_0}, \frac{dQ_r(u_0)}{du_0}$

We may now find the energy of the fluid motion. This is, the density of the fluid being unity,

$$T=-\int_0^{2\pi}2\pi\rho\frac{dn'}{dv}dv \, . \, \phi\frac{d\phi}{dn}$$

and

$$\frac{dn'}{dv}=\frac{a}{C-c}$$

$$\frac{d\phi}{dn}=V\frac{z}{r}=V\frac{Ss}{C-c}$$

$$\rho=\frac{aS}{C-c}$$

$$\therefore T=-2\pi a^2VS^2\int_0^{2\pi}\frac{s}{(C-c)^3}\phi dv$$

$$=-8\sqrt{2}a^3V^2S^2\Sigma A_nP_n\int_0^{2\pi}\frac{\sin v\sin nv}{(C-c)^{\frac{5}{2}}}dv$$

$$=-\frac{16\sqrt{2}}{3}a^3V^2S\Sigma A_nP_n\frac{d}{du}\left(\frac{1}{S}\frac{d}{du}\right)\int_0^{2\pi}\frac{\cos(n-1)v-\cos(n+1)v}{\sqrt{C-c}}dv$$

But

$$\int_0^{2\pi}\frac{\cos nv}{\sqrt{C-c}}dv=2\sqrt{2}Q_n$$

$$\therefore T=-\frac{64}{3}a^3V^2S\Sigma A_nP_n\frac{d}{du}\frac{1}{S}(Q'_{n-1}-Q'_{n+1})$$

But

$$Q'_{n-1}-Q'_{n+1}=-2nSQ_n$$

$$\therefore T=\frac{128}{3}a^3V^2S\Sigma_1^\infty nA_nP_nQ'_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

which is more convergent than the series for $\phi$.

In a similar manner may be found the velocity potential for any motion of translation, or the magnetism induced in a uniform field of force.

[*September*, 1881.—At the suggestion of one of the Referees I give a few additional numerical illustrations. The first is the ratio of the density of electricity at a point on a tore furthest from the axis to that at a point nearest the axis. The potential due to the distribution of electricity on the tore is given by (39). The normal force at any point of the tore is

$$\frac{\delta\phi}{\delta u}\frac{\delta u}{\delta n}=-\frac{C-c}{a}\frac{\delta\phi}{\delta u}$$

whilst for points furthest from the axis $u=u'$, $v=0$, and for points nearest $u=v'$, $v=\pi$. Putting these in, remembering that $2S dP_n/du=(2n+1)(P_{n+1}-CP_n)$ and $(2n+1)(P_{n+1}Q_n-P_nQ_{n+1})=2\pi$, it is easily shown that the above ratio is

$$=\left(\frac{C+1}{C-1}\right)^{\frac{3}{2}}\frac{2\pi\left(\frac{1}{2P_0}-\frac{1}{P_1}+\frac{1}{P_2}-\ldots\right)-\frac{1}{2}(Q_0+Q_1)-2(C+1)\Sigma(-)^n nQ_n}{2\pi\left(\frac{1}{2P_0}+\frac{1}{P_1}+\frac{1}{P_2}+\ldots\right)+\frac{1}{2}(Q_0-Q_1)-2(C-1)\Sigma nQ_n}$$

If the first $n$ sequence equations in Q be added together there results

$$4(C-1)\Sigma_0^n nQ_n=(2n+1)(Q_{n+1}-Q_n)+Q_0-Q_1$$

whence $4(C-1)\Sigma nQ_n=Q_0-Q_1$

Further, putting $(-)^nQ_n=q_n$, the sequence equation for $q$ is

$$(2n+1)q_{n+1}+4nCq_n+(2n-1)q_{n-1}=0$$

whence as before

$$4(C+1)\Sigma(-)^n nQ_n=4(C+1)\Sigma nq_n=q_1-q_0=-(Q_0+Q_1)$$

Finally then the ratio of the densities is

$$=\left(\frac{1+k'}{1-k'}\right)^3\frac{\frac{1}{2P_0}+\Sigma_1\frac{(-)^n}{P_n}}{\frac{1}{2P_0}+\Sigma_1\frac{1}{P_n}}$$

If terms higher than $P_1$ be neglected this is

$$=\left(\frac{1+k'}{1-k'}\right)^3\frac{E-2k'F}{E+2k'F}$$

I have not been able to find a finite expression for $\Sigma 1/P_n$ and $\Sigma(-)^n/P_n$, but when the ratio of $r$ to R is very small, the first two terms are sufficient. In any other case we can easily find the limits of error produced by neglecting terms after a given one. Thus suppose all after $P_r$ be neglected, then § 10

$$\frac{1}{P_n} > \frac{1}{C+S}\frac{1}{P_{n-1}} < \frac{1}{CP_{n-1}}$$

whence it follows that

$$\Sigma_{r+1}^{\infty} 1/P_n > \frac{k'}{1-k'}\frac{1}{P_r} < \frac{2k'}{(1-k')^2}\cdot\frac{1}{P_r}$$

Similarly it may be shown that, $r$ being odd

$$\Sigma_{r+1}^{\infty}(-)^n/P_n > \frac{k'}{1+k'}\frac{1-3k'-2k'^2}{(1-k')^2}\cdot\frac{1}{P_r}$$

$$< \frac{2k'}{1+k'^2}\left\{\left(\frac{1+k'^2}{1-k'^2}\right)^2 - \frac{k'}{1-k'^2}\right\}\frac{1}{P_r}$$

For the two cases of $k' = \sin 3°$ and $k' = \sin 6°$ (corresponding very nearly to $R=10r$ and $5r$ respectively) the ratios are ·5171 and ·2656.

The ratio of the velocity of the fluid at the centre of a tore to that of the tore itself when it moves without cyclic motion, parallel to its axis, is easily found. The point is given by $u=0$ $v=\pi$ which makes $P_n=\pi$. The velocity of the fluid $=\frac{\delta\phi}{\delta v}\frac{dv}{dn}$

$$=\frac{8V\sqrt{2}}{\pi}\sqrt{2}\Sigma_1^{\infty} n A_n \pi(-)^n$$

therefore ratio $=-16\Sigma_1^{\infty}(-)^n n A_n$

In the table below are given the values in two cases of $\alpha$, $A_1$, $A_2$, $T'$ (the effective mass of the fluid measured in terms of the fluid displaced), and $V'$, the ratio of the velocity at the centre, to that at an infinite distance when the tore is held at rest in the stream.

| $k'$ | $\alpha$ | $A_1$ | $A_2$ | $T'$ | $V'$ |
|---|---|---|---|---|---|
| sin 3° | −·00645 | −·00216 | −·00000 | ·99995 | 1·03456 |
| sin 6° | −·01868 | −·00871 | −·00007 | 1·09449 | 1·13712 |

Suppose the tore held in a uniform field of electric force parallel to its axis. The potential of the field is

$$\phi = \mu z$$

$$=\frac{4a\mu\sqrt{2}}{\pi}\sqrt{(C-c)}\Sigma n Q_n \sin nv$$

Hence, supposing the tore to be at zero potential and to have no charge, the potential of the disturbed field is, dashed letters denoting functions of $u'$,

$$=-\frac{4a\mu\sqrt{2}}{\pi}\sqrt{(C-c)}\Sigma n\left(Q_n-\frac{Q'_n}{P'_n}P_n\right)\sin nv$$

The density at any point of the tore is

$$=-\frac{\mu\sqrt{2}}{\pi^2}(C'-c)^{\frac{3}{2}}\Sigma\frac{n}{P'_n}\left\{P'_n\frac{dQ'_n}{du}-Q'_n\frac{dP'_n}{du}\right\}\sin nv$$

$$=\frac{\mu\sqrt{2}}{\pi S'}(C'-c)^{\frac{3}{2}}\Sigma\frac{n}{P'_n}\sin nv$$

Now at the points where the osculating plane touches the tore $z=r$ and $\rho=R$, whence

$$C-c=Ss \text{ or } c=1/C \qquad s=S/C$$

The greatest density on a sphere similarly influenced is $\frac{3\mu}{4\pi}$. The ratio is then

$$=\frac{4\sqrt{2}}{3}\frac{S^2}{C^{\frac{3}{2}}}\left\{\frac{S}{CP'_1}+\frac{4S}{C^2P'_2}+\right\}$$

The value of this ratio for the cases already considered are

for $k'=\sin 3°$, ·675

,, $k'=\sin 6°$, ·698

When the direction of the electric field is perpendicular to the axis, its potential is

$$\phi_1=\mu\rho\cos w=\mu a\frac{S\cos w}{C-c}$$

Hence clearly the functions for the expansion of this are the tesseral functions, $P_{1n}$, $Q_{1.n}$, and the conditions, since the potential holds for space outside the tore, are that

$$\mu a\frac{S\cos w}{C-c}+\sqrt{C-c}\cos w\Sigma A_nP_{1.n}\cos nv=0$$

when $u=u'$ for all values of $v$.

Hence

$$\pi A_n P'_{1.n} + 2\mu a S\int_0^\pi \frac{\cos nv}{(C-c)^{\frac{3}{2}}}\,dv = 0$$

or

$$\pi A_n P'_{1.n} - 4\mu a \frac{d}{du}(Q'_{0.n}\sqrt{2}) = 0$$

and

$$A_n = \frac{2\mu a\sqrt{2}}{\pi S'P'_{1.n}}(2n+1)(Q'_{0.n+1} - CQ'_{0.n})$$

But

$$A_0 = \frac{\mu a\sqrt{2}}{\mu S'P_{1.0}}(Q'_{0.1} - CQ'_{0.0})$$

From the first of (32*a*)

$$SP_{1.n} = P_{0.n+1} - CP_{0.n} = \frac{2S}{2n+1}\frac{d}{du}(P_{0.n})$$

which enables us to write the above in several ways. As before, the densities of electricity induced at points $(u'.0.0)$ and $(u'.\pi.0)$ are easily found.]